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Asymptotic analysis of some spectral problems in high contrast homogenisation and in thin domains

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

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Mikhail Cherdantsev
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Summary

We study the spectral properties of two problems involving small parameters. The first one is an eigenvalue problem for a divergence form elliptic operator $A_\varepsilon$ with high contrast periodic coefficients of period $\varepsilon$ in each coordinate, where $\varepsilon$ is a small parameter. The coefficients are perturbed on a bounded domain of ‘order one’ size. The local perturbation of coefficients for such operator could result in emergence of localised waves in the gaps of the Floquet-Bloch spectrum. We prove that, for the so-called double porosity type scaling, the eigenfunctions decay exponentially at infinity, uniformly in $\varepsilon$. Then, using the tools of two-scale convergence for high contrast homogenisation, we prove the strong two-scale convergence of the eigenfunctions of $A_\varepsilon$ to the eigenfunctions of a two-scale limit homogenised operator $A_0$, consequently establishing ‘asymptotic one-to-one correspondence’ between the eigenvalues and the eigenfunctions of these two operators. We also prove by direct means the stability of the essential spectrum of the homogenised operator with respect to the local perturbation of its coefficients. That allows us to establish not only the strong two-scale resolvent convergence of $A_\varepsilon$ to $A_0$ but also the Hausdorff convergence of the spectra of $A_\varepsilon$ to the spectrum of $A_0$, preserving the multiplicity of the isolated eigenvalues.

As the second problem we consider the eigenvalue problem for the Laplacian in a network of thin domains with Dirichlet boundary conditions. We construct an asymptotic solution to the problem using the method of matched asymptotic expansions to obtain appropriate boundary conditions for the terms of the asymptotics near the junctions of thin domains. We justify the asymptotics and prove the error bound of order $h^{3/2}$, where $h$ is the width of thin domains. We then derive a limiting model on the graph (which serves as a frame for such domain) and prove that it gives a proper approximation for the eigenvalues and eigenfunctions of the original problem. An important new result is that the boundary conditions at the vertices of the graph are mixed boundary conditions involving the small parameter $h$. This type of conditions keeps the information about the interaction between the edges of the graph and at the same time provides a better approximation than previously known models. We also study the bottom of the spectrum of the problem, whose corresponding eigenfunctions are confined to the vertices.
Introduction: motivation, literature overview, and structure of the thesis

The present thesis consists of two parts studying two separate problems: spectral convergence in homogenisation of high contrast media with a defect, and spectral asymptotics for networks of thin domains. Both themes are unified by the need to develop asymptotic analysis for associated spectral problems, employing relevant tools from asymptotic methods, spectral theory, non-classical homogenisation, etc. In turn, both topics are motivated by applications such as photonics and phononics and quantum graphs.

The motivation for the first part of the thesis arises, in particular, from recent growth of interest to photonic and phononic crystals and crystal fibers. The photonic (phononic) crystals are composite materials that often have a periodic structure. The fundamental property of the photonic (phononic) crystals consists in the existence of special regions (bandgaps) of frequencies where no electromagnetic (elastic) waves can propagate. Mathematically this regions correspond to the gaps in the essential spectrum of the related elliptic operators. This effect opens large possibilities for various applications in physics. In particular introduction of a defect in a periodic photonic or phononic fiber can lead to a spatial localisation of waves near the defect. While photonic applications come from optics (with problems described mathematically by Maxwell’s equation of electromagnetism) and phononic applications come from acoustics and elastodynamics, in both cases the key idea is that appropriate periodic media do not allow propagation of waves of certain frequency ranges. For example, the photonic crystal fibers, see e.g. [42], Figure 0–1, are typically represented by a core surrounded by a periodic cladding. Consequently, on the cross-section of the fiber the core itself represents a ‘defect’ with regards to the periodic cladding. In
addition, it is well known that, in general, the width of the photonic (phononic) bandgap (which is obviously an important property of a crystal) increases when the contrast between physical characteristics of components becomes larger.

Inspired by the above mentioned facts we study a simplified mathematical model of a photonic/phononic crystal, described by a divergence type operator with high contrast periodic coefficients with a finite size defect and the periodicity size $\varepsilon$ being a small parameter. (See [32] for a comprehensive review of mathematics of photonic crystals.)

There are several different mathematical aspects concerning the study of this sort of problems. First of all the above mentioned physical bandgap effect, in mathematical terms, is described by tools of spectral theory of differential operators with periodic coefficients, known as Floquet-Bloch theory. Namely, the above ‘forbidden’ frequencies precisely correspond to gaps in the spectra of such operators. Moreover, the emergence of localised modes due to defects in such periodic media corresponds in turn to eigenfunctions due to extra point spectrum appearing in the gaps. We hence first give a brief overview of the Floquet-Bloch theory, see e.g. [31], [41, v.4]. The Floquet-Bloch theory was originally developed by physicists to address problems involving periodic potentials, e.g. in
solid state physics. It was probably first realised by Gelfand [24] that in the multi-dimensional case it can be described by means of the spectral theory of self-adjoint operators. The key point here is the Floquet transform of a function $f : \mathbb{R}^n \to \mathbb{R}$, initially defined for $f \in C_0^\infty(\mathbb{R}^n)$, and then extended by continuity to $L^2(\mathbb{R}^n)$:

$$(Uf)(x,k) = \sum_{\xi \in \mathbb{Z}^n} f(x - \xi) e^{i k \cdot \xi}.$$ 

It has two important properties: quasi-periodicity with respect to $x$,

$$(Uf)(x + m, k) = e^{i k \cdot m} (Uf)(x,k), \quad \forall m \in \mathbb{Z}^n;$$

and periodicity with respect to $k$,

$$(Uf)(x, k + 2\pi m) = (Uf)(x,k), \quad \forall m \in \mathbb{Z}^n.$$ 

So, from considering a function defined on an unbounded domain ($\mathbb{R}^n$) one passes to considering a function of two variables defined on a bounded domain: $(x,k) \in Q \times Q^*$, where $Q = [0,1]^n$ is the periodic cell, and $Q^* = [0,2\pi]^n$ is the dual cell of 'quasimomenta' (the so-called Brillouin zone). Let us denote by $L(x,D)$ an elliptic differential operator $L(x,D)u = -\nabla \cdot A(x) \nabla u$, where $A(x)$ is a measurable periodic positive definite matrix, i.e. $\nu I \leq A(x) \leq \nu^{-1} I$ in the sense of quadratic forms for some $\nu > 0$, $A(x + m) = A(x)$, $\forall m \in \mathbb{Z}^n$. Due to its periodicity $L(x,D)$ commutes with the Floquet transform, i.e. for any $f \in C_0^\infty(\mathbb{R}^n)$

$$(U(Lf))(x,k) = L(x,D)(Uf)(x,k).$$

However now, on the right hand side of (2), for each $k$ the operator $L(x,D)$ acts in a different domain of functions satisfying quasi-periodicity condition (1).

So we have a family of operators $L(k)$ acting in spaces of functions defined on a compact domain. Hence each operator $L(k)$, appropriately extended to the self-adjoint one, has a discrete spectrum $\sigma(L(k)) = \bigcup_{j=1}^\infty \lambda_j(k)$. Then the following central spectral property can be shown for the spectrum $\sigma(L(x,D))$ of $L(x,D)$, see e.g. [31]:

$$\sigma(L(x,D)) = \bigcup_{k \in Q^*} \sigma(L(k)) = \bigcup_{j=1}^\infty \bigcup_{k \in Q^*} \lambda_j(k) = \bigcup_{j=1}^\infty B_j.$$ 

The spectrum of $L(x,D)$ has hence a band-gap structure: the bands $B_j, j \geq 1$, ...
may cease to overlap, resulting thereby in the presence of the gaps. Moreover, if the periodic coefficients of $L(x, D)$ are compactly perturbed, which physically corresponds to introduction of a defect, the spectral theory assures that the essential spectrum remains unperturbed, and hence the only extra spectrum can be the discrete spectrum in the gaps.

Therefore the spectral theory allows us to connect mathematical objects (e.g. band-gaps and point spectrum in the gaps) with physical effects (e.g. forbidden frequencies and localised modes). However, problems of the existence of the gaps, their location and width, the existence of point spectrum due to defects, the properties of the related eigenfunctions etc, have no general answer and require additional analytical or numerical investigation. Our key idea is to advance in these directions analytically using asymptotic methods, i.e. exploiting the presence of a small parameter. In our context, $\varepsilon$ describes the size of the periodicity, which is the standard setup of the homogenisation theory being reviewed next.

In the presence of a small parameter one normally looks for some asymptotic approximation to the problem. Namely, periodic rapidly oscillating problems are usually treated by the means of well developed theory of homogenisation, which was originated as mathematical discipline probably in the work of De Giorgi and Spagnolo. The idea of homogenisation is to approximate a given operator by some homogenised operator (with constant or slowly varying coefficients). There are several different approaches to this theory, which often can supplement each other. One uses the method of asymptotic expansions, which assumes that the solution to an appropriate differential equation

$$L_\varepsilon u_\varepsilon := -\nabla \cdot A(x/\varepsilon) \nabla u_\varepsilon = f$$

(3)

can be represented in the form

$$u_\varepsilon = u_0(x, \varepsilon^{-1}x) + \varepsilon u_1(x, \varepsilon^{-1}x) + \varepsilon^2 u_2(x, \varepsilon^{-1}x) + \ldots ,$$

(4)

where the terms are assumed to be periodic in the second variable, $u_i(x, y + m) = u_i(x, y)$, $m \in \mathbb{Z}^n$, $i = 0, 1, \ldots$. Substituting this ansatz into the equation and

---

1We do not discuss in this thesis the issue of whether `embedded` eigenvalues can emerge on the bands as a result of the perturbation.

2Note that there are other ways of applying asymptotics methods in the present context, not using homogenisation, see e.g. [26, 35].

3Condition of periodicity (on the $\varepsilon$-scale) can be relaxed in various ways or removed altogether, see e.g [27, 47], which we do not address in this thesis.
equating the coefficients at same powers of $\varepsilon$ one arrives at a recurrent sequence of equations depending on two variables $x$ and $y = \varepsilon^{-1}x$. One first observes that $u_0$ is function of $x$ only, $u_0(x, y) = u_0(x)$. Then the corrector $u_1(x, y)$ is found in the form

$$u_1(x, y) = \sum_{k=1}^{n} N_k(y) \frac{\partial u_0}{\partial x_k},$$

where $N_k \in H^1_{\text{loc}}(\mathbb{R}^n)$ is a periodic solution of ‘unit cell’ problems

$$-\nabla_y \cdot A(y) \nabla_y N_k(y) = -\sum_{i=1}^{n} \frac{\partial}{\partial y_i} a_{ik}(y),$$

($a_{ij}$ are entries of the matrix $A$). Finally, the solvability condition for $u_2$ leads to the homogenised equation for $u_0$:

$$-\nabla \cdot A^\text{hom} \nabla u_0 = f,$$

where $A^\text{hom}$ is the homogenised matrix of constant coefficients given by

$$A^\text{hom} = \int_Q A(y)(I + \nabla_y N) dy.$$

Here $I$ is the unity matrix and $\nabla_y N$ is the matrix with columns $\nabla_y N_1$, $\nabla_y N_2$, ..., $\nabla_y N_n$.

The problem of justification, or convergence of $u_\varepsilon$ to $u_0$, has received considerable separate attention. The above procedure of asymptotic expansion can be advanced further, using the uniform ellipticity of $L_\varepsilon$, to obtain not only the convergence but also error bounds establishing the rate of convergence. For instance for bounded $\Omega$ with Lipschitz boundary $\partial\Omega$ and Dirichlet boundary conditions on $\partial\Omega$ one has $\|u_\varepsilon - (u_0 + \varepsilon u_1(x, x/\varepsilon))\|_{H^1(\Omega)} \leq C\varepsilon^{1/2}$, see e.g. [5, 10, 27, 43] and further references therein. An alternative method of directly passing to the limit is based on selecting appropriate oscillatory test functions in the weak formulation of (3), using methods of compensated compactness, see e.g. [37, 47].

Another approach to homogenisation is associated with the method of two-scale convergence. The idea of the two-scale convergence is to preserve in the limit the information about oscillations of elements of a sequence on $\varepsilon$ scale. For example, in the sense of the usual convergence in $L^2$-norm a sequence $f(x) \sin(\varepsilon^{-1}x)$ weakly converges to zero (and there is no strong convergence). However, in the sense of two-scale convergence this sequence strongly converges to the function of
two variables $f(x)\sin(y)$. In general, the strong two-scale convergence of $u_\varepsilon(x)$ to $u_0(x,y)$, $u_\varepsilon(x) \rightarrow^2 u_0(x,y)$, loosely means $\|u_\varepsilon(x) - u_0(x,x/\varepsilon)\|_{L^2(\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$. The first and crucial step in this direction was made by Nguetseng in [38] where he actually introduced (what we now call) weak two-scale convergence, proved the weak two-scale compactness of a sequence of bounded in $L^2$-norm functions (which is a two-scale analogue of the Banach-Alaoglu theorem), and derived a formula for the weak two-scale limit of gradients of a bounded in $H^1$-norm sequence of functions. He applied these results to the homogenisation of a periodic uniformly elliptic problem obtaining classical results and also a new convergence formula for the gradient of the solution of the problem. Later Allaire [2], relying on the work of Nguetseng, introduced the notion of strong two-scale convergence and developed further the theory and its applications to some problems of homogenisation for operators with periodic coefficients. The theory of two-scale convergence was advanced thereafter, among others, by Zhikov [48], see below, who in particular extended it for (periodic) measures and applied it to study the convergence of spectra, see also [50].

Classical homogenisation is incapable of accounting for the above described effects: the homogenised operator has constant coefficients and therefore its spectrum is the whole positive semi-axis with no band gaps. However, certain versions of non-classical homogenisation do allow one to account for some of these effects, as we discuss next. On the other hand, the method of two-scale convergence is applicable not only to the above reviewed ‘classical’ homogenisation, but also to various ‘non-classical’ versions of homogenisation. The non-classical homogenisation includes higher-order homogenisation, see e.g. [5, 20, 46], exponential homogenisation [28], non-local homogenisation [9, 15, 17, 18, 19, 21], etc., which often refer to high contrast in the coefficients (see also [45]), as is the case in our model. (Notice that, as additional motivation for our simplified model, in e.g. photonic crystal fibers even low-contrast structures can sometimes display ‘apparent’ high contrast on a cross-section, see e.g. [11] for a physical discussion.)

Following e.g. [2] and [48] we consider a special scaling of the coefficients $a(x,\varepsilon)$ of the operator $-\nabla \cdot a(x,\varepsilon)\nabla$.

Namely, we let $a(x,\varepsilon)$ be of order 1 in the matrix phase and of order $\delta$ in the inclusion phase. The asymptotic behaviour will then crucially depend on the relation between the two small parameters $\varepsilon$ and $\delta$, a phenomenon known in physical literature as ‘noncommuting limits’, see e.g. [40] and further references.
therein. It is well-known that $\delta(\varepsilon) \sim \varepsilon^2$ is a critical scaling in this context, often referred to as a ‘double porosity-type’ scaling. This is the only scaling that leads in the limit to the dependence on the fast variable $y$ (i.e. the main order term $u_0(x,y)$ in the asymptotic expansion (4) retains the dependence on $y$, whereas other scalings of order $\varepsilon^\alpha$ where $\alpha \neq 2$ lead to either the classical homogenised problem with no dependence on $y$ ($\alpha < 2$) or to a degenerate problem ($\alpha > 2$), see [2] and [48] for more detailed explanations. The problem of high contrast (or double porosity-type) homogenisation has become a popular subject in the past two decades, in particular, it was firstly treated by the two-scale convergence method in [2]. Since the principal interest of our study is in the spectral characteristics of the problem, we mainly refer to the two works by Zhikov [48, 49], where the author, in particular, developed further the method of two-scale convergence, including its application to the high contrast homogenisation, described the spectrum of the limit homogenised operator in an explicit way and proved convergence of the spectra of the periodic operators to the spectrum of the homogenised one in the sense of Hausdorff (see Section 1.1). The spectrum of the homogenised operator has an explicit band-gap structure, hence so does the spectrum of the periodic operator for small enough $\varepsilon$.

As it was mentioned earlier, due to the presence of gaps in the spectrum of the operator one can expect that an introduction of a defect into the periodic structure of the coefficients may lead to emergence of localised modes, i.e. eigenvalues in the gaps of the essential spectrum with corresponding eigenfunctions concentrated near the defect. Indeed, it was proven in [23] that for a given gap in the spectrum of a periodic operator one can introduce a defect in the periodic media, i.e. can perturb locally the coefficients of the operator, so that the perturbed operator will have at least one eigenvalue inside the gap. Moreover, as was also proven in [23], under the compact perturbation of coefficients the essential spectrum of the operator remains unperturbed and the eigenfunctions corresponding to the eigenvalues in the gaps decay exponentially at infinity. This type of results is actually quite general in the perturbation theory of self-adjoint operators (see e.g. [12, 41]).

We now describe our problem and the results in more detail. In the first chapter we study an elliptic divergence form operator $A_\varepsilon$ with locally perturbed

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4The term ‘double porosity’ originates from mathematically similar problems of fluid flows in fractured porous rocks [7].
5This relates asymptotically to phenomenon of ‘micro-resonances’, both in phononic and photonic contexts, found in physical literature, see e.g. [33, 34].
high contrast (of order $\varepsilon^2$) $\varepsilon$-periodic coefficients. The behaviour of $A_\varepsilon$ and its spectral characteristics as $\varepsilon \to 0$ is the main topic of interest. A similar problem is considered in [29] using the method of asymptotic expansions, but the present study pursues different aims and approaches the problem from another direction, namely developing an appropriate version of the two-scale convergence technique [2, 38, 49]. As a result we obtain a complete description of the asymptotic (with respect to $\varepsilon$) behaviour of the localised modes and other spectral characteristics for the operator $A_\varepsilon$ in terms of an explicitly described (two-scale) limit operator $A_0$. For other recent applications of the high contrast homogenisation techniques see also [4, 8, 13, 16, 19, 21, 30, 44].

In the absence of a defect, Zhikov considers in [48] a divergence form elliptic operator $\hat{A}_\varepsilon$ (denoted in [48] by $A_\varepsilon$) with periodic coefficients corresponding to a double-porosity model [3, 14] ($A_\varepsilon$ in our notation is obtained from $\hat{A}_\varepsilon$ by a compact perturbation of its coefficients). Operators of such type have the Floquet-Bloch essential spectrum, displaying a band-gap structure. Zhikov proves that the spectra of $\hat{A}_\varepsilon$ converge in the sense of Hausdorff to the spectrum of a certain two-scale homogenised operator $\hat{A}_0$ with constant coefficients, see also [26, 49], and that $\hat{A}_0$ is the limit of $\hat{A}_\varepsilon$ in the sense of strong two-scale resolvent convergence. The spectrum of $\hat{A}_0$ is purely essential and displays an explicit band-gap structure. As we already mentioned, in the case of a compact perturbation of periodic coefficients in the elliptic operator $\hat{A}_\varepsilon$ its essential spectrum remains unperturbed, see e.g. [23, 41]. The only extra spectrum that can emerge in the gaps due to the perturbation is a discrete one (isolated eigenvalues with finite multiplicity). Such an extra spectrum does emerge at least under some assumptions, e.g. [23, 29]. This corresponds physically to localised modes emerging near the defect.

One of the main goals of the first part of the thesis is to establish the strong two-scale convergence of the eigenfunctions of $A_\varepsilon$ corresponding to the eigenvalues in the gaps. In order to achieve this we need the strong two-scale compactness of the eigenfunctions. This requires in turn an exponential decay of the eigenfunctions uniform in $\varepsilon$.

The problem of wave localisation (i.e. of the existence of eigenvalues with corresponding eigenfunctions decaying exponentially) in the gaps of the essential spectrum has been intensively investigated for a wide range of differential operators over the last decades. The results obtained up to date ensure the exponential decay of eigenfunctions of $A_\varepsilon$ for a fixed $\varepsilon$, see e.g. [23]. However
this is insufficient for establishing the required compactness. Moreover, the developed methods, e.g. [6] and [23] (the latter using the method of Agmon[1]), seem to be insufficient for the present purpose. The reason is that in order to obtain the \textit{uniform} exponential decay one has to perform some kind of two-scale asymptotic analysis, investigating the behaviour of the eigenfunctions on small and large scales simultaneously. To achieve this we supplement the method of [1] by the related two-scale techniques, which play a crucial role. As a result, we obtain a uniform estimate with the decay exponent \( \alpha \) (see (1.24) and (1.13) below) which ensures the compactness, but may also be of an independent interest. On one hand, it is sharp in a sense as \( \varepsilon \to 0 \). On the other hand, it behaves qualitatively entirely differently compared to e.g. the one in [6]: while the one in [6] is proportional to the square root of the distance to the gap end, the decay exponent we derive becomes large on approaching the left end of the gap and small near the right end.

The structure of the first part is the following. We first define the problem in Section 1.1, describe the two-scale limit operator \( A_0 \) and state the main result. We then consider a subsequence of eigenvalues of \( A_\varepsilon \) converging to some point \( \lambda_0 \) lying in a gap of the spectrum of \( \hat{A}_0 \). In Section 1.2 we prove (Theorem 1.2.2) the uniform exponential decay for the eigenfunctions of \( A_\varepsilon \). Section 1.3 is devoted to the proof of a main auxiliary lemma that is employed in the previous section, which may also be of an independent interest. In Section 2.1 we list some properties of the two-scale convergence and several related statements which we use in the next section. Employing the uniform exponential decay, we establish in Section 2.2 (see Theorem 2.2.1) the strong two-scale compactness of (normalised) eigenfunctions of \( A_\varepsilon \), see e.g. [48, 49]. This implies that, up to a subsequence, the eigenfunctions two-scale converge to a function, which is eventually proved to be an eigenfunction of the two-scale limit operator \( A_0 \) with a defect, which could be considered as a perturbation of \( \hat{A}_0 \). Accordingly \( \lambda_0 \) is an eigenvalue of \( A_0 \). The two-scale convergence of the eigenfunctions together with the results of [29] on the existence of the eigenvalues in the gaps and related error bounds allow us to make a conclusion about the ‘asymptotic one-to-one correspondence’ between eigenfunctions and eigenvalues of the operators \( A_\varepsilon \) and \( A_0 \) as \( \varepsilon \to 0 \). In the Section 2.3 we prove by direct means (via Weyl sequences) the stability of the essential spectrum of \( \hat{A}_0 \) with respect to the local perturbation of its coefficients (see Theorem 2.3.1). Thereby this establishes the convergence of the spectra of \( A_\varepsilon \) to the spectrum of \( A_0 \) in the sense of Hausdorff (Theorem 1.1.1).
Another interesting topic in asymptotic analysis relates to various problems in thin structures which naturally arise in physics, chemistry, engineering, when one considers for instance propagation of waves in a network of thin domains. When the cross-sectional size of such an object is much smaller than its length it is natural to try to approximate the original problem by a differential (or sometimes more general) equation on a graph eliminating the transversal dimensions. In this case one obtains a so-called “quantum graph”, i.e. one-dimensional differential equation posed on the graph. Probably one of the first quantum graph models was employed in chemistry where one considered a model of free electron motion along a carcass of a molecule (see e.g. [79]). Other examples can be found in nanotechnology and mesoscopic physics where several dimensions of physical objects are reduced to a size of a few nanometers [54]. Problems in thin domains appear in many other areas of mathematics and have been studied in different contexts, see e.g. [52, 53, 55, 56, 57, 61, 63, 75].

First we recall some results obtained for models related to graphs with straight edges. Consider a domain $\Omega_h$ given as an $h$-neighbourhood of a planar graph, where $h$ is a small parameter, see Figure 0-2. Let $A_h$ be an elliptic self-adjoint differential operator in $\Omega_h$ with some boundary conditions. We are interested in the spectrum of such operator. It is natural to try reducing the given problem in $\Omega_h$ to some problem on the graph. In the limit as $h$ tends to zero one normally obtains a differential operator $A$, e.g. $A = -\frac{\partial^2}{\partial s^2}$, where $s$ is an arclength along the edges, which must be equipped with appropriate boundary conditions at vertices. The latter is not always a trivial question, and for some boundary value problems on $\Omega_h$ it is still (or was until recently) essentially open, see e.g. [63] or [71] for the relevant discussion.
The case of Neumann boundary conditions is probably the easiest one. In this case the first transversal eigenvalue is zero and bounded states confined to the junctions of $\Omega_h$ are absent. The limiting operator is equipped with the so-called Kirchhoff boundary conditions at each vertex $v_l$:

$$\sum_{\{j|v_l \in e_j\}} \frac{df}{dx_j}(v_l) = 0.$$ 

Here $e_j$ denotes edges of the graph. The following result in a more or less general way was obtained by different researchers (see e.g. [59, 70, 77, 78, 80]):

$$\lambda_n(A_h) \to \lambda_n(A_0) \text{ as } h \to 0,$$

where $\lambda_n$ is the $n$-th eigenvalue of a corresponding operator in the ascending order, counted with multiplicities. The idea of the proof is the following: using the minimax definition of the eigenvalues one needs to construct a mapping from the $H^1$ space on graph into the $H^1$ space on $\Omega_h$ and vice versa such that the ratio $\frac{\|\nabla f\|_2}{\|f\|_2}$ does not increase substantially.

The case of the Dirichlet Laplacian is considerably more difficult. There are two reasons for that. The first one is that the spectrum of the Dirichlet Laplacian behaves completely differently compared to the Neumann Laplacian case. The first transversal eigenvalue $\nu_0$ for Dirichlet boundary conditions is non-zero. Hence the corresponding eigenvalues of $A_h$ should behave essentially as $h^{-2}\nu_0$. Additionally there may be bounded states living below the part of the spectrum generated by the transversal eigenvalues. Another difficulty lies in finding appropriate conditions at vertices of the graph for the limiting problem, as was already mentioned.

A classical and very popular tool for dealing with problems in graph-like domains is the method of matched asymptotic expansions. Employing this method one considers an inner problem in a neighbourhood of a junction and an outer problem in an adjacent strip and then one must match the corresponding solutions in some intermediate region in the vicinity of the junction.

Consider the outer problem, i.e. eigenvalue problem for the Dirichlet Laplacian in a thin (of width $h$) strip. One can introduce a stretched transversal variable $\eta = h^{-1}y$ so that the problem is considered in a fixed rectangle. Then
the Laplacian in the new variables reads

$$- \Delta = -h^{-2} \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial s^2}. \quad (5)$$

One can then seek a solution to the outer problem in the separation of variables form

$$u_h(s, \eta) = v_i(s) \varphi_j(\eta), \quad \lambda_h = h^{-2} \nu_j + \mu_i, \quad (6)$$

where $\nu_j$ and $\varphi_j$ are the transversal eigenvalues and eigenfunctions satisfying Dirichlet boundary conditions and $\mu_i$ and $v_i$ are eigenvalues and eigenfunctions of the operator $-\frac{d^2}{dx^2}$ which is not fully defined due to uncertain boundary conditions at the vertices of the graph. We restrict ourself to considering the eigenvalues of $A_h$ generated by the first transversal eigenvalue $\nu_0$.

The inner problem is set in the ‘spider domain’ $\Pi$ obtained from the rescaled (by $h^{-1}$) neighbourhood of a junction by attaching straight strips $\Pi_j$ of infinite length and width 1, see Figure 0-3. In order to match a solution of the inner problem with the solution of outer problem one needs to consider the following equation

$$-\Delta g = (\nu_0 + h^2 \mu_i) g \text{ in } \Pi,$$

$$g = 0 \text{ on } \partial \Pi.$$
introduce local coordinates \((x, y)\), so that \(y\) is the transversal coordinate. A function \(g = g_p\) is called a solution of the scattering problem in \(\Pi\) if it has the following asymptotic behaviour in each infinite strip \(\Pi_j, j = 1, \ldots, m:\)

\[
g_p = \delta_{pj} e^{-ih\sqrt{\mu_i}x} \varphi_0(y) + s_{pj} e^{ih\sqrt{\mu_i}x} \varphi_0(y) + O(e^{-\beta x}),
\]

(7)

where \(\beta > 0\) is some constant (which depends on \(h\) and \(\mu_i\)), \(\delta_{pj}\) is the Kronecker symbol and \(\varphi_0\) is transversal eigenfunction corresponding to \(\nu_0\) (in our case \(\varphi_0(y) = \sin(\pi y)\)). The first term in (7) can be interpreted as an incident wave coming from infinity along the strip \(\Pi_p\) and all the remaining terms describe the transmitted (including reflected, \(j = p\)) waves. The matrix

\[
S = \begin{bmatrix} s_{pj} \end{bmatrix}
\]

is called the scattering matrix. \(S\) is unitary and depends on \(h\) analytically.

Matching the asymptotics of the inner and the outer solutions, one can obtain a description of the spectrum of \(\Omega_h\) in terms of spectrum of the operator \(-\frac{d^2}{dx^2}\) acting on the graph with some boundary conditions (gluing conditions) at the vertices which depends on the scattering matrix. This program was carried out in recent works \([64, 72, 73, 74]\).

The existence of bound states in strip-like domains is well known from the waveguide theory, see e.g. \([58, 60]\) (see also \([68, 76]\) for the similar effect in a thin plates), where bounded states are proven to exist in \(L\)-shaped domains or as induced by a curvature. Thus, below the part of the spectrum induced by the transversal eigenvalues there can exist eigenvalues of \(\Omega_h\) corresponding to the bound states with eigenfunctions confined to the junctions of \(\Omega_h\) or ‘sharp’ bends of its channels. Apparently, these eigenvalues cannot be described in terms of the limiting operator on the graph.

In the present work we study a spectral problem for the negative Dirichlet Laplacian in a simplified graph-like domain \(\hat{\Omega}_h\) with non-straight strips. Our main goal is to obtain a delicate asymptotic description of the spectrum of \(\hat{\Lambda}_h\) in terms of limiting operators on the graph. One can start with considering a symmetric graph that consists of only two edges joining in a single vertex at an angle less than \(\pi\). We assume that the edges are straight in some neighbourhood of the vertex. The corresponding \(\hat{\Omega}_h\) is symmetric with respect to the bisectrix of the angle between edges of the graph, see Figure [0-4]. This implies that the eigenfunctions of \(\hat{\Lambda}_h\) satisfy either Dirichlet or Neumann boundary conditions.
at the bisectrix. Then it is sufficient to consider the eigenvalue problem for the negative Laplacian in \( \Omega_h \), which is a part of \( \hat{\Omega}_h \) lying on one side of the bisectrix, with Dirichlet or Neumann boundary conditions at the bisectrix (being now a slanted end of \( \Omega_h \)) and Dirichlet conditions elsewhere. The limiting graph for \( \Omega_h \) in this case is a simple curve. The case of Dirichlet boundary conditions on the slanted end of \( \Omega_h \) is very similar to the Neumann one and yet is simpler. So we consider only the Neumann case, and denote the corresponding operator by \( A_h \), see the precise definitions and illustrations in Chapter 3.

We implement the general scheme of matched asymptotic expansions outlined above. Considering the outer problem we change the variables so as to flatten the domain and scale the transversal variable by \( h^{-1} \). The main terms of the asymptotic solution the eigenvalue problem in the new domain have form (6), however \( \mu_i, \nu_i \) solve now the eigenvalue problem for the operator \(-\frac{d^2}{dx^2} - \frac{1}{4}\kappa^2\), where \( \kappa \neq 0 \). Hence some eigenvalues \( \mu_i \) can be negative (which is different from already studied problems in [64, 72, 73, 74]). We construct further terms of the asymptotics to obtain more accurate approximations to the eigenelements of \( A_h \) (namely, the error bound of order \( h^{3/2} \) is proven).

Matching the asymptotics of the outer solution with the asymptotics of the scattering solution of the inner problem we derive the boundary conditions for the limiting operator. The scattering matrix \( S \) (which is merely a complex number in our setting of the problem) depends analytically on \( h \). We use its asymptotic expansion obtained in [66], which is given in terms of the scattering
matrix corresponding to the threshold case

\[-\Delta g = \nu_0 g.\]

Normally, one has Dirichlet boundary conditions for \( v_i \) at the end of the curve corresponding to the slanted end of \( \Omega_h \). However for some geometric configurations (i.e. for some values of the angle of the slant) a Neumann boundary condition is possible. Namely, this is the case when the scattering solution at the threshold is bounded in \( L^\infty \)-norm (i.e. being the so-called generalised bounded state); we call this the critical case.

We also consider the eigenvalues corresponding to the bounded states in the semi-infinite straight strip obtained from \( \Omega_h \) by rescaling it in the neighbourhood of the slanted end. It is well known that there exists at least one bounded state lying below the transversal eigenvalue \( \nu_0 \), see [69]. We provide some new estimates on the number of such bounded states with respect to the value of the angle of the slant.

In the case of Dirichlet boundary conditions on the slanted end of \( \Omega \) the limiting problem always has Dirichlet boundary conditions at the corresponding end of the curve. There do not exist bounded states for the rescaled semi-infinite strip in this case.

The structure of the second part is the following. In Chapter 3 we construct the asymptotics of the problem in \( \Omega_h \). We state the problem in Section 3.1. In Section 3.2 we derive a formal asymptotic solution to the outer problem. In the next section we recall some results on scattering solutions of the inner problem from [66], also deriving order \( h \) term in the asymptotics of the scattering matrix in the critical case. Then we match the asymptotics of the inner and the outer solutions and consequently obtain the boundary conditions for auxiliary problems on the limiting graph for the terms of the asymptotic expansion in Section 3.4. Section 3.5 is devoted to the justification of the asymptotics. Chapter 4 is devoted to the construction of the limiting model graph, which is probably the most important result of the second part. In the last two sections we study properties of the bottom of the spectrum of the operator \( A_h \), which is related to the bound states in the rescaled semi-infinite strip lying below the first transversal eigenvalue \( \nu_0 \). Notice that the notation that we use in Part II may be different from the one used in the present introduction.
Part I

Spectral convergence for high contrast media with a defect via homogenisation
Chapter 1

Uniform exponential decay of eigenfunctions

In this chapter we first formulate the problem studied in the first part of the thesis. We then review already known results related to our problem, namely, the results on spectral convergence for the high contrast homogenisation when there is no defect \[48, 49\], and on existence of the eigenvalues of the operator $A_\varepsilon$ near the eigenvalues of the limit homogenised operator $A_0$ \[29\]. We also describe the structure of the homogenised operators and properties of their spectra. In the end of Section 1.1 we state the main result of the chapter on the exponential decay uniform with respect to $\varepsilon$ of the eigenfunctions of $A_\varepsilon$, which subsequently implies the two-scale compactness of the sequence of eigenvalues (Chapter 2). The rest of the chapter is devoted to the proof of the uniform exponential decay.

1.1 Notation, problem formulation, limit operator and the main result

We will use the following notation for the geometric configuration visualised on Figure 1-1, cf. \[29\]. Consider a periodic set of unit cubes

\[\{Q : Q = [0, 1)^n + \xi, \xi \in \mathbb{Z}^n\}. \tag{1.1}\]

Let $F_0$ be an open periodic set with period one in each coordinate such that $F_0 \cap Q \subset Q$ is a connected domain with infinitely smooth boundary. We denote $F_0 \cap Q$ by $Q_0$ and its complement $Q \setminus Q_0$ by $Q_1$. Notice that the position of the
particular set $Q_0, Q_1$ or $Q$ depends on $\xi \in \mathbb{Z}^n$, however we will not reflect this in the notation to simplify the latter. Regularity assumptions on the boundary could be relaxed. Let $\Omega_2$ be a bounded domain containing the origin and with a sufficiently smooth boundary; its complement is denoted by $\Omega_1$, $\Omega_1 = \mathbb{R}^n \setminus \Omega_2$.

We define the ‘inclusion phase’ or the ‘soft phase’ $\Omega_0^\varepsilon$ as

$$\Omega_0^\varepsilon = \bigcup_{\varepsilon Q_0 \subset \Omega_1} \varepsilon Q_0,$$

where $\varepsilon > 0$ is a small parameter. The set of inclusions $\varepsilon Q_0$ which intersect the boundary of $\Omega_2$ is denoted by $\tilde{\Omega}_0^\varepsilon$. The ‘matrix phase’, denoted by $\Omega_1^\varepsilon$, is the complement to the inclusions in $\Omega_1$, i.e. $\Omega_1^\varepsilon = \Omega_1 \setminus (\Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon)$. ‘Defect domain’ $\Omega_2^\varepsilon$ is defined by $\Omega_2 \setminus \tilde{\Omega}_0^\varepsilon$. We also use the notation $\Theta_{\Omega}$ for the characteristic function of a set $\Omega$ and $B_R$ for the open ball of radius $R$ centred at the origin.

We consider an eigenvalue problem

$$A_\varepsilon u^\varepsilon = \lambda_\varepsilon u^\varepsilon$$

(1.2)
for the point spectrum of an elliptic operator $A_\varepsilon$, self-adjoint in $L^2(\mathbb{R}^n)$,

$$A_\varepsilon u^\varepsilon := -\nabla \cdot \left( a(x,\varepsilon) \nabla u^\varepsilon(x) \right), \ x \in \mathbb{R}^n,$$

(1.3)

with coefficient $a(x,\varepsilon)$ given by the formula

$$a(x,\varepsilon) = \begin{cases} 
  a_0 \varepsilon^2, & x \in \Omega^\varepsilon_0, \\
  a_1, & x \in \Omega^\varepsilon_1, \\
  a_2, & x \in \Omega^\varepsilon_2, \\
  \tilde{a}_0(x,\varepsilon), & x \in \tilde{\Omega}^\varepsilon_0, 
\end{cases}$$

(1.4)

where measurable $\tilde{a}_0(x,\varepsilon)$ is such that

either $\tilde{A}_0 \varepsilon^{2-\theta} \leq \tilde{a}_0(x,\varepsilon) \leq \sigma_0 \varepsilon^{2-\theta}$ for all $\varepsilon$, or $\tilde{a}_0(x,\varepsilon) = a_0 \varepsilon^2$ for all $\varepsilon$. (1.5)

Here $a_0$, $a_1$, $a_2$, $\tilde{A}_0$, $\sigma_0$ and $\theta$ are some positive constants independent of $\varepsilon$, $\theta \in (0,2]$. Notice that this includes as particular cases e.g. the case of ‘removed’ boundary inclusions, i.e. $a(x,\varepsilon) = a_1$ if $x \in \tilde{\Omega}^\varepsilon_0 \cap \Omega_1$, $a(x,\varepsilon) = a_2$ if $x \in \tilde{\Omega}^\varepsilon_0 \cap \Omega_2$, and the case of the ‘full’ inclusions, $\tilde{a}_0(x,\varepsilon) = a_0 \varepsilon^2$. The domain of $A_\varepsilon$ is defined in a standard way via Friedrichs extension procedure with a bilinear form

$$B_\varepsilon(u,w) = \int_{\mathbb{R}^n} a(x,\varepsilon) \nabla u \cdot \nabla w \, dx$$

defined on $H^1(\mathbb{R}^n)$. By definition, $u^\varepsilon \in H^1(\mathbb{R}^n)$, $u^\varepsilon \not\equiv 0$, is an eigenfunction of the eigenvalue problem (1.2) with an eigenvalue $\lambda_\varepsilon$ if

$$B_\varepsilon(u^\varepsilon,w) = \lambda_\varepsilon \int_{\mathbb{R}^n} u^\varepsilon w \, dx$$

(1.6)

for all $w \in H^1(\mathbb{R}^n)$.

Properties of $A_\varepsilon$ are closely associated with properties of a corresponding purely periodic high contrast self-adjoint operator $\hat{A}_\varepsilon$, i.e. with no defect present. The operator $\hat{A}_\varepsilon$ is generated (via Friedrichs extension procedure) by a bilinear form

$$\hat{B}_\varepsilon(u,w) = \int_{\mathbb{R}^n} \hat{a}(x,\varepsilon) \nabla u \cdot \nabla w \, dx$$

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acting on $H^1(\mathbb{R}^n)$ with a coefficient

$$\tilde{a}(x, \varepsilon) = \begin{cases} 
  a_0\varepsilon^2, & x \in \varepsilon F_0, \\
  a_1, & x \in \varepsilon(\mathbb{R}^n \setminus F_0).
\end{cases}$$

It is well known that the spectrum of a periodic operator is so called Floquet-Bloch spectrum, it is purely essential and has a band-gap structure. This operator was considered by Zhikov in [48, 49]. He proves that the spectra of $\hat{A}_\varepsilon$ converge in the sense of Hausdorff to the spectrum of a certain homogenised operator $\hat{A}_0$.

By definition, the Hausdorff convergence of spectra, $\sigma(\hat{A}_\varepsilon) \xrightarrow{H} \sigma(\hat{A}_0)$ as $\varepsilon \to 0$, means that

- for any $\lambda \in \sigma(\hat{A}_0)$ there exists a sequence $\lambda_\varepsilon \in \sigma(\hat{A}_\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$;
- if $\lambda_\varepsilon \in \sigma(\hat{A}_\varepsilon)$ and $\lambda_\varepsilon \to \lambda$, then $\lambda \in \sigma(\hat{A}_0)$.

The limiting operator $\hat{A}_0$ is of a ‘two-scale’ nature. It acts in a Hilbert space

$$\hat{H}_0 := \left\{ u(x, y) \in L^2(\mathbb{R}^n \times Q) \left| u(x, y) = u_0(x) + v(x, y), 
\begin{array}{l}
u_0 \in L^2(\mathbb{R}^n), v \in L^2(\mathbb{R}^n; L^2(Q_0))
\end{array}\right. \right\},$$

with the natural inner product inherited from $L^2(\mathbb{R}^n \times Q)$ and $\hat{H}_0$ being its closed subspace. At this point we suppose for definiteness that $Q = [0, 1)^n$. It is implied that $v$ is extended by zero for $y \in Q_1$. In what follows we will assume that a function defined for $y \in Q$ is extended by periodicity to the whole $\mathbb{R}^n$.

The operator $\hat{A}_0$ is defined as generated by a (closed) symmetric and bounded from below bilinear form $\hat{B}_0(u, w)$ acting in a dense subspace

$$\hat{V} = H^1(\mathbb{R}^n) + L^2(\mathbb{R}^n, H^1_0(Q_0))$$

of $\hat{H}_0 = L^2(\mathbb{R}^n) + L^2(\mathbb{R}^n, L^2(Q_0))$, which is defined as follows: for $u = u_0 + v, w = w_0 + z \in \mathcal{V}$,

$$\hat{B}_0(u, w) = \int_{\mathbb{R}^n} A_{\text{hom}} \nabla u_0 \cdot \nabla w_0 \, dx + a_0 \int_{Q_0} \int_{\mathbb{R}^n} \nabla_y v \cdot \nabla_y z \, dy \, dx. \quad (1.9)$$

Here $A_{\text{hom}} = (A_{ij}^{\text{hom}})$ is the standard “porous” homogenised (symmetric, positive-definite) matrix for the periodic medium as described above but when no defect

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is present and with \( a_0 = 0 \), see e.g. \[27, \S 3.1\]:

\[
A_{ij}^{\text{hom}} \xi_i \xi_j = \inf_{w \in C^\infty_{\text{per}}(Q)} \int_{Q_1} a_1 |\xi + \nabla w|^2 \, dy \quad (\xi \in \mathbb{R}^n).
\]

(1.10)

Notation \( C^\infty_{\text{per}}(Q) \) stands for the set of infinitely smooth functions with periodic boundary conditions. Then one can see that the form is indeed bounded from below, densely defined and closed. Hence, according to the standard Friedrichs extension procedure, e.g. \[41\], \( \widehat{A}_0 \) can be defined as a self-adjoint operator with a domain \( D(\widehat{A}_0) \subset \widehat{V} \).

It is also proved in \[49\] that the spectrum of \( \widehat{A}_0 \) is purely essential and has a band-gap structure. It can be described in terms of a function \( \beta(\lambda) \) which we introduce next, see \[48, 49\] (cf. also \[13\]). First we define an operator \( T \) as follows,

\[
Tf := -a_0 \Delta f, \quad f(y) \in H_0^1(Q_0) \cap H^2(Q_0).
\]

(1.11)

Denote by \( b \) the solution to

\[
Tb - \lambda b = -a_0 \Delta b - \lambda b = 1, \quad b(y) \in H_0^1(Q_0).
\]

(1.12)

Then the function \( \beta(\lambda) \) is defined by the formula

\[
\beta(\lambda) := \lambda(1 + \lambda\langle b \rangle_y),
\]

(1.13)

where \( \langle f \rangle_y := \int_Q f(y) \, dy \) denotes a mean value of a function in a unit cell.

One can get a more transparent notion of \( \beta(\lambda) \) by applying a spectral decomposition to problem (1.12). Let \( \lambda_i, \lambda'_j \) and \( \varphi_i, \varphi'_j, \ i, j = 1, 2, \ldots \), be all eigenvalues (repeated accordingly to their multiplicity) and corresponding orthonormalised eigenfunctions of \( T \), where eigenfunctions \( \varphi'_j \) have zero mean, \( \langle \varphi'_j \rangle_y = 0 \). The set of eigenfunctions of \( T \) makes a basis in \( H_0^1(Q_0) \). Hence we can write \( b \) as

\[
b = \sum_{i=1}^{\infty} c_i \varphi_i + \sum_{j=1}^{\infty} c'_j \varphi'_j.
\]

We substitute this expansion into (1.12) to obtain

\[
\sum_{i=1}^{\infty} (\lambda_i - \lambda) c_i \varphi_i + \sum_{j=1}^{\infty} (\lambda'_j - \lambda) c'_j \varphi'_j = 1.
\]
Multiplying the latter by \( \varphi_i \) or \( \varphi_j' \) and integrating we arrive at

\[
    c_i = \frac{\langle \varphi_i \rangle_y}{\lambda_i - \lambda}, \quad c_j' = 0.
\]

This yields us the following expression for \( \beta(\lambda) \),

\[
    \beta(\lambda) = \lambda + \lambda^2 \sum_{i=1}^{\infty} \frac{\langle \varphi_i \rangle_y^2}{\lambda_i - \lambda}, \quad (1.14)
\]

see Figure 1-2. The intervals where \( \beta(\lambda) \geq 0 \) correspond to the bands of the spectrum of \( \hat{A}_0 \). Isolated points of the spectrum of \( \hat{A}_0 \), i.e. \( \lambda_j' \) such that \( \beta(\lambda_j') < 0 \), can also be regarded as degenerate bands. The intervals on which \( \beta(\lambda) < 0 \) (excluding \( \lambda_j' \)) are gaps.

The operator \( A_\varepsilon \) is obtained from \( \hat{A}_\varepsilon \) by a compact perturbation of its coefficient. It was shown in [23] (cf. also [11]) that in this case the essential spectrum of \( A_\varepsilon \) coincides with the spectrum of \( \hat{A}_\varepsilon \) and only extra eigenvalues can emerge, in particular in the gaps. We do not consider possible emergence of embedded eigenvalues, i.e. eigenvalues in the bands of essential spectrum. Existence of embedded eigenvalues is believed to be very unlikely, but this supposition has not been proved. In this work we are interested in convergence properties of the eigenvalues of \( A_\varepsilon \) lying in the gaps of its spectrum and the corresponding eigenfunctions. We will prove that if a sequence of eigenvalues converges to a point lying in the gap of \( \sigma_{\text{ess}}(\hat{A}_0) \), then the latter is an eigenvalue of the 'limit' homogenised operator \( A_0 \). The operator \( A_0 \) can be obtained from \( \hat{A}_0 \) by a compact perturbation of the coefficients. Its definition, analogous to the definition of
\( \hat{A}_0 \), is the following. The operator \( A_0 \) acts in a Hilbert space

\[
\mathcal{H}_0 := \left\{ u(x, y) \in L^2(\mathbb{R}^n \times Q) \left| u(x, y) = u_0(x) + \nu(x, y), \quad u_0 \in L^2(\mathbb{R}^n), \nu \in L^2(\Omega_1; L^2(Q_0)) \right. \right\},
\]

(1.15)

It is defined via a Friedrichs extension procedure by a closed symmetric and bounded from below bilinear form

\[
B_0(u, w) = a_2 \int_{\Omega_2} \nabla u_0 \cdot \nabla w_0 \, dx + \int_{\Omega_1} A^{\text{hom}} \nabla u_0 \cdot \nabla w_0 \, dx + a_0 \int_{\Omega_1} \int_{Q_0} \nabla_y \nu \cdot \nabla_y z \, dy \, dx
\]

(1.16)

acting in a dense subspace

\[
\mathcal{V} = H^1(\mathbb{R}^n) + L^2(\Omega_1, H^1_0(Q_0))
\]

(1.17)

of \( \mathcal{H}_0 = L^2(\mathbb{R}^n) + L^2(\Omega_1, L^2(Q_0)) \), \( u = u_0 + \nu, w = w_0 + z \in \mathcal{V} \). By definition \( \lambda_0 \) is an eigenvalue of \( A_0 \) and \( u^0(x, y) = u_0(x) + \nu(x, y) \in \mathcal{V} \) is corresponding eigenfunction if

\[
B_0(u^0, w) = \lambda_0(u^0, w)_{\mathcal{H}_0},
\]

(1.18)

for any \( w = w_0 + z \in \mathcal{V} \). The eigenfunction solves the following problem:

\[
-\nabla \cdot a_2 \nabla u_0(x) = \lambda_0 u_0(x), \quad x \in \Omega_2,
\]

\[
-\nabla \cdot A^{\text{hom}} \nabla u_0(x) = \lambda_0 (u_0 + \langle \nu \rangle_y), \quad x \in \mathbb{R}^n \setminus \Omega_2,
\]

\[
-a_0 \Delta_y \nu = \lambda_0 (u_0 + \nu), \quad y \in Q_0; \quad \nu = 0, \ y \in \partial Q_0 \ (x \in \mathbb{R}^n \setminus \Omega_2),
\]

\[
(u_0)_- = (u_0)_+ + a_2 \left( \frac{\partial u_0}{\partial n} \right)_- = \left( \sum_{i,j} A^{\text{hom}}_{ij} \frac{\partial u_0}{\partial x_j} n_i \right)_+, \quad x \in \partial \Omega_2.
\]

(1.19)

Here

\[
\langle \nu \rangle_y(x) := |Q|^{-1} \int_Q \nu(x, y) \, dy
\]

denotes the averaging with respect to \( y \) over the periodicity cell \( Q \) (extending \( \nu \) by zero outside \( Q_0 \)); \( (\cdot)_- \) and \( (\cdot)_+ \) denote respectively the interior and exterior limit values of the appropriate entities at the boundary \( \partial \Omega_2 \) of \( \Omega_2 \), \( n \) is the interior unit normal to \( \partial \Omega_2 \).

A similar problem is considered in [29]. The authors use an asymptotic ex-
pansion approach seeking a solution to problem (1.2) in the form

$$
\begin{align*}
    u^\varepsilon(x) &= u^0(x, x/\varepsilon) + \varepsilon u^{(1)}(x, x/\varepsilon) + \varepsilon^2 u^{(2)}(x, x/\varepsilon) + \ldots, \\
    \lambda(\varepsilon) &= \lambda_0 + o(1).
\end{align*}
$$

They prove that if there exists an eigenvalue $\lambda_0$ satisfying $\beta(\lambda_0) < 0$, $\lambda_0 \neq \lambda_j'$, then there exists $\varepsilon_0 > 0$ and a constant $C_1 > 0$ independent of $\varepsilon$ such that for any $0 < \varepsilon \leq \varepsilon_0$ there exists an isolated eigenvalue $\lambda_\varepsilon$ of operator $A_\varepsilon$ of finite multiplicity, such that

$$
|\lambda_\varepsilon - \lambda_0| < C_1 \varepsilon^{1/2}. \quad (1.20)
$$

Moreover if $(u_0, v)$ is an eigenfunction of $A_0$ which corresponds to $\lambda_0$ then the function

$$
\begin{align*}
    u^{\text{appr}}(x, \varepsilon) := \begin{cases} \\
    u_0(x) + v(x, x/\varepsilon), & x \in \Omega_\varepsilon^c, \\
    u_0(x), & x \in \Omega_1^c \cup \Omega_2 \cup \tilde{\Omega}_0^c,
    \end{cases} \quad (1.21)
\end{align*}
$$

is an approximate eigenfunction for $A_\varepsilon$ at least in the following sense: there exist constants $c_j(\varepsilon)$ such that

$$
\|u^{\text{appr}} - \sum_{j \in J_\varepsilon} c_j(\varepsilon) u^\varepsilon_j\|_{L^2(\mathbb{R}^n)} < C_2 \varepsilon^{1/2}, \quad (1.22)
$$

where $J_\varepsilon = \{ j : |\lambda_{\varepsilon, j} - \lambda_0| < C \varepsilon^{1/2} \}$ is a finite set of indices (for each $\varepsilon$), and $\lambda_{\varepsilon, j}$, $u^\varepsilon_j(x)$ are eigenvalues and $L_2$-normalised eigenfunctions of $A_\varepsilon$, and the constants $C_1$ and $C_2$ are independent of $\varepsilon$.

This assertion partly answers the problem of asymptotic behaviour of the discrete spectrum of $A_\varepsilon$. Thus, we know that any eigenvalue of $A_0$ has converging to it a sequence of eigenvalues of $A_\varepsilon$. In this work we study an open question that consists in the following. Suppose there is a sequence of eigenvalues of $A_\varepsilon$ converging to a point in the gap of $\sigma(\tilde{A}_0)$, $\lambda_\varepsilon \to \lambda_0$. Is the limit $\lambda_0$ an eigen-value of $A_0$ or not? To answer this question affirmatively one firstly needs to show a compactness (in the sense of two-scale convergence) of the corresponding eigenfunctions. Once having the compactness proved one can pass to a limit in the spectral problem (1.2) to get eventually the spectral problem for the homogenised operator. In its turn the proof of compactness requires uniform with respect to $\varepsilon$ exponential decay at infinity of eigenfunctions $u_\varepsilon$ corresponding to a convergent sequence of eigenvalues.
Now we formulate our main result.

**Theorem 1.1.1.** The operator $A_\varepsilon$ converges to $A_0$ in the sense of the strong two-scale resolvent convergence. Hence the spectral projectors also strongly two-scale converge away from the point spectrum of $A_0$. The spectrum of $A_\varepsilon$ converges in the sense of Hausdorff to the spectrum of $A_0$. Let $\lambda_0$ be an isolated eigenvalue of multiplicity $m$ of the operator $A_0$ in the gap of its essential spectrum. Then, for small enough $\varepsilon$, there exist exactly $m$ eigenvalues $\lambda_{\varepsilon,i}$ of $A_\varepsilon$ (counted with their multiplicities) such that

$$|\lambda_{\varepsilon,i} - \lambda_0| \leq C_\varepsilon^{1/2}, \; i = 1, \ldots, m,$$

(1.23) with a constant $C$ independent of $\varepsilon$. If for some sequence $\varepsilon_k \to 0$ a sequence of eigenvalues $\lambda_{\varepsilon_k}$ of $A_\varepsilon$ converges to $\lambda_0$ which is in the gap of the essential spectrum of $A_0$, then $\lambda_0$ is an isolated eigenvalue of $A_0$ of a finite multiplicity $m$ and for large enough $k$, $\lambda_{\varepsilon_k} \in \{\lambda_{\varepsilon,i}, i = 1, \ldots, m\}$.

### 1.2 Uniform exponential decay of the eigenfunctions of $A_\varepsilon$

The phenomenon of exponential decay of eigenfunctions of various differential operators corresponding to the eigenvalues in the gaps of essential spectra has been extensively investigated for the few last decades, see e.g. [6, 22, 23, 39]. For example, in [6] a Schrödinger operator $H$ with random perturbation is considered. It is shown that the rate of the exponential decay is proportional to $\sqrt{\Delta_+(E)\Delta_-(E)}$, where $E$ is an eigenvalue in a gap, $\Delta_+(E)$ and $\Delta_-(E)$ are distances from $E$ to the right and left edges of the gap respectively. Roughly speaking, the exponent obtained in [6] is proportional to the distance from the essential spectrum near the centre of the gap and to the square root from the distance near the both edges of the gap. This estimate is the best of the sort known at present. Another result, obtained in [23], can be straightforwardly applied to the operator $A_\varepsilon$ when $\varepsilon$ is fixed. It follows from [23] that an eigenfunction $u_\varepsilon$ decays exponentially with an exponent proportional to $\text{dist}(\lambda_{\varepsilon}, \sigma_{\text{ess}}(A_\varepsilon))$. Nevertheless, this is not sufficient for our purposes. To gain compactness of a sequence

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2The error bound (1.23) employs the results of [29] requiring, as stated, higher regularity of $\partial Q_0$. The rest of the statement of the theorem applies potentially to less regular boundaries.
we need uniform decay, i.e. exponential decay with exponent independent of \( \varepsilon \), which previous results do not guarantee. In this section we prove \( \varepsilon \)-uniform exponential decay of sequence of \( u^\varepsilon \) corresponding to converging in the gap \( \lambda_\varepsilon \).

**Remark 1.2.1.** We would like to draw attention to the qualitative difference between the previously obtained estimates for the rate of exponential decay and the one we prove in this work. As we mentioned above, known results give the rate of decay proportional to the distance to gap edges or to the square root of the distance. Our estimate (1.24) is entirely different. For small enough \( \varepsilon \) the rate of decay is \( O \left( \text{dist} (\lambda_\varepsilon, \sigma_{\text{ess}}(A_\varepsilon))^{1/2} \right) \) at the right end of each interval \( \beta(\lambda) < 0 \) and proportional to \( (\text{dist} (\lambda_\varepsilon, \sigma_{\text{ess}}(A_\varepsilon))^{-1/2} \) at the left end, cf. Figure 1-2 and (1.14).

We formulate the main result of this section (and also one of the principal results of the first part) in the following

**Theorem 1.2.2.** Let \( \lambda_{\varepsilon_k} \) and \( u^\varepsilon_k \) be sequences of eigenvalues of the operator \( A_\varepsilon \) and corresponding eigenfunctions normalised in \( L^2(\mathbb{R}^n) \), where \( \varepsilon_k \) is some positive sequence converging to zero as \( k \to \infty \). Let \( \lambda_0 \) be such that \( \beta(\lambda_0) \) is negative and \( \lambda_0 \) is not an eigenvalue of the operator \( T \) given by (1.11). Suppose that \( \lambda_{\varepsilon_k} \) converges to \( \lambda_0 \). Then for small enough \( \varepsilon_k \) eigenfunctions \( u^\varepsilon_k \) decay uniformly exponentially at infinity, namely, for

\[
0 < \alpha < \sqrt{-\beta(\lambda_0)/a_1}
\]

(1.24)

the following holds:

\[ \| e^{\alpha|x|} u^\varepsilon_k \|_{L^2(\mathbb{R}^n)} \leq C, \]

uniformly in \( \varepsilon_k \), i.e. for any \( 0 < \varepsilon_k < \varepsilon(\alpha) \), with \( C = C(\alpha) \) independent of \( \varepsilon \).

**Proof.** We drop the index \( k \) in \( \varepsilon_k \) for the sake of simplification of notation. So, when we say, for instance, ‘sequence \( \lambda_\varepsilon \)’ we actually mean ‘subsequence \( \lambda_{\varepsilon_k} \)’.

The plan of the proof is the following. We first derive ‘elementary’ a priori estimates for the eigenfunction \( u^\varepsilon \) outside the set of inclusions \( \Omega_0 \cup \tilde{\Omega}_0 \). Next we study the structure of the eigenfunction at the small scale and deduce some vital inequalities for \( \varepsilon \nabla u^\varepsilon \) inside the inclusions. As a central technical step, we then employ in the integral identity (1.6) a test function with exponentially growing weight \( g^2(|x|) \), see (1.37)–(1.38) below, and perform some delicate two-scale uniform estimates to achieve the result. Introducing a test function with exponentially growing weight we use the idea of Agmon, [1]. This on its own does
not lead to a straightforward conclusion. We have to develop some delicate two-scale analysis, studying the properties of eigenfunctions of $A_\varepsilon$ both at large and small scales at the same time. The main auxiliary technical results are proven in Lemma 1.2.4 and Proposition 1.3.1.

**Step 1.** Due to the nature of the operator $A_\varepsilon$ (coefficient $a(x,\varepsilon)$ is very small on the inclusion phase) one can expect that the eigenfunctions (more precise, their gradients) oscillate wildly on the inclusion phase. Nevertheless it is possible to control $u_\varepsilon$ outside the inclusions. Setting $w = u_\varepsilon$ in (1.6) we have

$$
\varepsilon^2 a_0 \| \nabla u_\varepsilon \|_{L^2(\Omega_0^0)}^2 + a_1 \| \nabla u_\varepsilon \|_{L^2(\Omega_1^0)}^2 + a_2 \| \nabla u_\varepsilon \|_{L^2(\Omega_2^0)}^2 +
$$

$$
+ \| a_0^{1/2}(x,\varepsilon) \nabla u_\varepsilon \|_{L^2(\tilde{\Omega}_0^0)}^2 = \lambda_\varepsilon \| u_\varepsilon \|_{L^2(\mathbb{R}^n)}^2 = \lambda_\varepsilon.
$$

(1.25)

Therefore

$$
\| u_\varepsilon \|_{H^1(\mathbb{R}^n \setminus (\Omega_1 \cup \tilde{\Omega}_0^0))} \leq C
$$

(1.26)

uniformly in $\varepsilon$. From now on $C$ denotes a generic constant whose precise value is insignificant and can change from line to line.

**Step 2.** Now we will represent $u_\varepsilon$ as a sum of two functions, one of them has $\varepsilon$-uniformly bonded norm in $H^1$, another preserves the ‘uncontrollable’ oscillations of the gradient of $u_\varepsilon$. Let us consider $u_\varepsilon$ in a cell $\varepsilon Q$ corresponding to such $\xi = \xi(\varepsilon) \in \mathbb{Z}^n$, see 1.1, that the respective ‘inclusion’ $\varepsilon Q_0$ has a nonempty intersection with $\Omega_1$. There exists an extension $\tilde{u}_\varepsilon$ of $u_\varepsilon|_{\epsilon Q_1}$ to the whole cell $\varepsilon Q$ such that

$$
\| \tilde{u}_\varepsilon \|_{L^2(\varepsilon Q_0)} \leq C \| u_\varepsilon \|_{L^2(\varepsilon Q_1)}, \quad \| \nabla \tilde{u}_\varepsilon \|_{L^2(\varepsilon Q_0)} \leq C \| \nabla u_\varepsilon \|_{L^2(\varepsilon Q_1)},
$$

(1.27)

where $C$ does not depend on $\varepsilon$ or $\xi$, see e. g. [35, Ch. 3, §4, Th. 1], which is a version of the so-called ‘extension lemma’, see also e.g. [27, §3.1, L. 3.2]. In particular, we can choose the following extension:

$$
\tilde{u}_\varepsilon \equiv u_\varepsilon, \quad x \in \Omega_1^0 \cup \tilde{\Omega}_0^0,
$$

$$
- \nabla \cdot (a(x,\varepsilon) \nabla \tilde{u}_\varepsilon(x)) = 0, \quad x \in \Omega_0^0 \cup \tilde{\Omega}_0^0,
$$

which minimises $\| a^{1/2}(x,\varepsilon) \nabla \tilde{u}_\varepsilon \|_{L^2(\varepsilon Q_0)}$ subject to the prescribed boundary conditions, with (1.4) and (1.5) ensuring that (1.27) still holds. From (1.26) and (1.27) we conclude that

$$
\| \tilde{u}_\varepsilon \|_{H^1(\mathbb{R}^n)} \leq C.
$$

(1.28)

31
We represent $u^\varepsilon$ in the form

$$u^\varepsilon(x) = \tilde{u}^\varepsilon(x) + v^\varepsilon(x)$$  \hspace{1cm} (1.29)

and consider the function $v^\varepsilon \in H^1_0(\Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon)$. We assume that $v^\varepsilon$ is extended by zero to the whole $\mathbb{R}^n$. In each inclusion $\varepsilon Q_0 \subset \Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon$ we have the following boundary value problem for $v^\varepsilon(x)$:

$$-\nabla \cdot (a(x, \varepsilon) \nabla v^\varepsilon) - \lambda_\varepsilon v^\varepsilon = \lambda_\varepsilon \tilde{u}^\varepsilon, \ x \in \varepsilon Q_0,$$

$$v^\varepsilon(x) = 0, \ x \in \partial(\varepsilon Q_0).$$  \hspace{1cm} (1.30)

When $a(x, \varepsilon) = a_0\varepsilon^2$, i.e. everywhere in $\Omega_0^\varepsilon$ and in $\tilde{\Omega}_0^\varepsilon$ in the case $\tilde{a}_0(x, \varepsilon) = a_0\varepsilon^2$, after changing the variables $x \to y = x/\varepsilon$ we obtain

$$-a_0 \Delta_y v^\varepsilon(\varepsilon y) - \lambda_\varepsilon v^\varepsilon(\varepsilon y) = \lambda_\varepsilon \tilde{u}^\varepsilon(\varepsilon y), \ y \in Q_0,$$

$$v^\varepsilon(\varepsilon y) = 0, \ y \in \partial Q_0.$$  \hspace{1cm} (1.31)

Since $\lambda_0 \neq \lambda_j$ by the assumptions of the theorem, $\lambda_\varepsilon$ is separated uniformly from the spectrum of the operator $T$, (1.11), for small enough $\varepsilon$. Hence the resolvent of $T$ at $\lambda_\varepsilon$ is bounded uniformly in $\varepsilon$ and (1.31) implies

$$\|v^\varepsilon(\varepsilon y)\|_{H^1(Q_0)} \leq C\|\tilde{u}^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}.$$  \hspace{1cm} (1.32)

In the case when $\tilde{A}_0 \varepsilon^{2-\theta} \leq \tilde{a}_0(x, \varepsilon) \leq \sigma_0 \varepsilon^{2-\theta}$, $\theta \in (0, 2]$, we multiply equation (1.30) by $v^\varepsilon$ and integrate by parts to obtain after rescaling

$$\varepsilon^{-2} \int_{Q_0} \tilde{a}_0(\varepsilon y, \varepsilon) |\nabla_y v^\varepsilon(\varepsilon y)|^2 dy - \lambda_\varepsilon \int_{Q_0} (v^\varepsilon(\varepsilon y))^2 dy = \lambda_\varepsilon \int_{Q_0} \tilde{u}^\varepsilon(\varepsilon y) v^\varepsilon(\varepsilon y) dy.$$  \hspace{1cm} (1.33)

Notice that $\varepsilon^{-2} \tilde{a}_0(\varepsilon y, \varepsilon) \geq \tilde{A}_0 \varepsilon^{-\theta} \to \infty$ as $\varepsilon \to 0$. Then using Poincaré inequality for functions from $H^1(Q_0)$

$$\int_{Q_0} f^2 dy \leq C \int_{Q_0} |\nabla_y f|^2 dy,$$
and Hölder inequality

$$\left[ \int_{Q_0} fg \, dy \right]^2 \leq \int_{Q_0} f^2 \, dy \int_{Q_0} g^2 \, dy,$$

we derive from (1.33) that

$$(C \varepsilon^{-\theta} - \lambda \varepsilon) \|v(\varepsilon y)\|_{L^2(Q_0)}^2 \leq \lambda \varepsilon \|v(\varepsilon y)\|_{L^2(Q_0)} \|\tilde{u}(\varepsilon y)\|_{L^2(Q_0)}.$$

The latter immediately implies that

$$\|v(\varepsilon y)\|_{L^2(Q_0)} \leq C \|\tilde{u}(\varepsilon y)\|_{L^2(Q_0)}.$$

In fact an even stronger relation is valid,

$$\|v(\varepsilon y)\|_{L^2(Q_0)} = o \left( \|\tilde{u}(\varepsilon y)\|_{L^2(Q_0)} \right).$$

From (1.33) and (1.34) one directly obtains

$$\varepsilon^{-2} \|\tilde{a}_0^{1/2} \nabla_y v(\varepsilon y)\|_{L^2(Q_0)} \leq C \|\tilde{u}(\varepsilon y)\|_{L^2(Q_0)}^2;$$

for small enough $\varepsilon$. Returning in (1.32) and in (1.34), (1.35) to the variable $x$ we arrive at the following inequality that describes the behaviour of $v(\varepsilon)$ and its gradient in $\Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon$,

$$\|a^{1/2} \nabla v(\varepsilon)(x)\|_{L^2(\varepsilon Q_0)} + \|v(\varepsilon)(x)\|_{L^2(\varepsilon Q_0)} \leq C \|\tilde{u}(\varepsilon)(x)\|_{L^2(\varepsilon Q_0)}^2,$$

with an $\varepsilon$-independent constant $C$.

**Step 3.** In order to get the uniform exponential decay of the eigenfunctions we next substitute in (1.6) a test function of a special form:

$$w = g^2(|x|) \tilde{u}(\varepsilon)(x).$$

Here we define function $g$ as follows

$$g(t) = \begin{cases} 
\alpha t, & t \in [0, R], \\
\alpha R, & t \in (R, +\infty),
\end{cases}$$

where $R$ is some arbitrary positive number. The exponent $\alpha$ will be chosen later. This method was employed e.g. by Agmon, see [1], but in the present case its realisation is not straightforward. Namely, to obtain the desired estimates we
have to implement the approach of [1] in the context of the two-scale analysis. We will show that $g(|x|)\tilde{u}^\varepsilon(x)$, and consequently $g(|x|)u^\varepsilon(x)$, are bounded in $L^2(\mathbb{R}^n)$ uniformly with respect to $R$ and $\varepsilon$. Then we will show via passing to the limit as $R \to \infty$ that we can replace $g(|x|)$ by $e^{\alpha|x|}$.

**Remark 1.2.3.** We cannot use $e^{2\alpha|x|}\tilde{u}^\varepsilon(x)$ as a test function directly, since it is not known at this stage that functions $e^{\alpha|x|}\tilde{u}^\varepsilon(x)$ and $e^{\alpha|x|}u^\varepsilon(x)$ are square integrable.

The following identity holds by direct inspection

$$\nabla \tilde{u}^\varepsilon \nabla (g^2 \tilde{u}^\varepsilon) = |\nabla (g\tilde{u}^\varepsilon)|^2 - |\nabla g|^2(\tilde{u}^\varepsilon)^2. \tag{1.39}$$

Notice that the Euclidian norm of $\nabla g$ is bounded by $g$ with constant $\alpha$ (uniformly with respect to $R$):

$$|\nabla g(|x|)| \leq \alpha g(|x|). \tag{1.40}$$

After the substitution of $(1.37)$ into $(1.6)$ we have, via $(1.29)$ and $(1.39)$,

$$\varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla u^\varepsilon \cdot \nabla (g^2 \tilde{u}^\varepsilon) \, dx + \int_{\Omega_0^\varepsilon} \tilde{a}_0 \nabla v^\varepsilon \cdot \nabla (g^2 \tilde{u}^\varepsilon) \, dx + \int_{\mathbb{R}^n \setminus \Omega_0^\varepsilon} a(x, \varepsilon)|\nabla (g\tilde{u}^\varepsilon)|^2 \, dx -$$

$$-a_1 \int_{\Omega_1^\varepsilon} |\nabla g|^2(\tilde{u}^\varepsilon)^2 \, dx - \lambda_\varepsilon \int_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} g^2(\tilde{u}^\varepsilon)^2 \, dx - \lambda_\varepsilon \int_{\Omega_0^\varepsilon} g^2 v^\varepsilon \tilde{u}^\varepsilon \, dx =$$

$$= \lambda_\varepsilon \int_{\Omega_0^\varepsilon} g^2 u^\varepsilon \tilde{u}^\varepsilon \, dx + \lambda_\varepsilon \int_{\Omega_1^\varepsilon} g^2(\tilde{u}^\varepsilon)^2 \, dx + \int_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} a(x, \varepsilon)|\nabla g|^2(\tilde{u}^\varepsilon)^2 \, dx. \tag{1.41}$$

Notice that the right hand side is bounded by some constant $C$ independent of $\varepsilon$ and $R$ due to $(1.26)$, $(1.28)$, $(1.36)$ and the boundedness of the domains of integration.

The rough idea of the remaining part of the proof is the following. We argue that the second term on the left hand side of the latter is small and the first one is relatively small (compared to the other terms of the identity). One can notice that in equation $(1.31)$ the right hand side is ‘almost’ a constant for every fixed $Q_0$. Then one can expect that the solution of $(1.31)$ is ‘approximately equal’ to
the solution of (1.12) corresponding to $\lambda = \lambda_\varepsilon$ multiplied by $\lambda_\varepsilon \tilde{u}^\varepsilon(\varepsilon y)$,

$$v^\varepsilon(\varepsilon y) \sim \lambda_\varepsilon b(y) \tilde{u}^\varepsilon(\varepsilon y).$$

Rearranging the integrated entities in the last two terms on the left hand side of (1.41) one obtain ‘approximately’

$$- \lambda_\varepsilon g^2(\tilde{u}^\varepsilon)^2 - \lambda_\varepsilon g^2 v^\varepsilon \tilde{u}^\varepsilon \sim - [\lambda_\varepsilon (1 + \lambda_\varepsilon b(y))] g^2(\tilde{u}^\varepsilon(\varepsilon y))^2. \tag{1.42}$$

The expression in the square brackets resembles the definition of $\beta(\lambda_\varepsilon)$, see (1.13). Rescaling back to variable $x$ and integrating the right hand side of (1.42) one can obtain

$$- \beta(\lambda_\varepsilon) \|g \tilde{u}^\varepsilon\|^2_{L^2(\tilde{\Omega}_0^\varepsilon \cup \tilde{\Omega}_1^\varepsilon)}. \tag{1.43}$$

Notice that as $\lambda_\varepsilon \to \lambda_0$, $\beta(\lambda_\varepsilon) \to \beta(\lambda_0) < 0$. Hence we obtain $\|g \tilde{u}^\varepsilon\|^2_{L^2(\tilde{\Omega}_0^\varepsilon \cup \tilde{\Omega}_1^\varepsilon)}$ multiplied by a uniformly positive coefficient, end we need only to choose appropriate exponent $\alpha$, see (1.38), to ensure that the fourth term on the left hand side of (1.41),

$$- a_1 \int_{\tilde{\Omega}_1^\varepsilon} |\nabla g|^2(\tilde{u}^\varepsilon)^2 \, dx > - a_1 \alpha^2 \|g \tilde{u}^\varepsilon\|^2_{L^2(\tilde{\Omega}_1^\varepsilon)},$$

is compensated by (1.43).

Let us continue the proof. Consider the second term on the left hand side of (1.41). Since the coefficient $\tilde{a}_0(x, \varepsilon)$ is bounded uniformly in $\varepsilon$ and the sequence of domains $\tilde{\Omega}_0^\varepsilon$ is also bounded (so $g^2|\tilde{\Omega}_0^\varepsilon$, $\nabla g^2|\tilde{\Omega}_0^\varepsilon \leq C$ uniformly) we derive that

$$\left| \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0 \nabla v^\varepsilon \cdot \nabla (g^2 \tilde{u}^\varepsilon) \, dx \right| \leq C \int_{\tilde{\Omega}_0^\varepsilon} a_0^{1/2} |\nabla v^\varepsilon| (|\nabla \tilde{u}^\varepsilon| + |\tilde{u}^\varepsilon|) \, dx \leq C \|a_0^{1/2} \nabla v^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} (\|\nabla \tilde{u}^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} + \|\tilde{u}^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)}).$$

Then from (1.28), (1.36) follows that

$$\left| \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0 \nabla v^\varepsilon \cdot \nabla (g^2 \tilde{u}^\varepsilon) \, dx \right| \leq C \|\tilde{u}^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)}.$$

The right hand side of the latter converges to zero. Indeed, let us take an arbitrary
subsequence $\tilde{u}^\varepsilon$. Since $\|\tilde{u}^\varepsilon\|_{H^1(\mathbb{R}^n)}$ is bounded uniformly in $\varepsilon$, see (1.28), the set of functions $\tilde{u}^\varepsilon$ is weakly compact in $H^1(B_R)$, hence strongly compact in $L^2(B_R)$ for any $R$; we take $R$ large enough so that $\Omega_2 \Subset B_R$. Then there exists a further subsequence $\tilde{u}^\varepsilon$ that converges to some function $u^\varepsilon$ strongly in $L^2(B_R)$. Then

$$
\|\tilde{u}^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} \leq \|u^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} + \|\tilde{u}^\varepsilon - u^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} \to 0
$$

(1.44)
as the Lebesgue measure of the set $\tilde{\Omega}_0^\varepsilon$ tends to zero. Since we have chosen in the beginning an arbitrary subsequence $\tilde{u}^\varepsilon$, (1.44) holds for any sequence of $\varepsilon$. Hence

$$
\left| \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0 \nabla v^\varepsilon \cdot \nabla (g^2 \tilde{u}^\varepsilon) \, dx \right| \to 0.
$$

(1.45)
From (1.36) and (1.44) we also obtain

$$
\|v^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} \to 0.
$$

(1.46)

**Step 4.** The following Lemma approximates the last two terms and bounds the first term (both, in a sense, of a ‘two-scale’ nature) on the left hand side of (1.41).

**Lemma 1.2.4.** There exists $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$ the following estimates are valid

$$
\left| \frac{\lambda_2}{\Omega_0 \cup \Omega_1} \int_{\Omega_0 \cup \Omega_1} g^2(\tilde{u}^\varepsilon)^2 \, dx + \frac{\lambda_2}{\tilde{\Omega}_0^\varepsilon} \int_{\tilde{\Omega}_0^\varepsilon} g^2 v^\varepsilon \tilde{u}^\varepsilon \, dx - \beta(\lambda_2) \frac{\lambda_2}{\Omega_0 \cup \Omega_1} \int_{\Omega_0 \cup \Omega_1} g^2(\tilde{u}^\varepsilon)^2 \, dx \right| \leq C \varepsilon \left( \|\nabla (g\tilde{u}^\varepsilon)\|_{L^2(\Omega_1)}^2 + \|g\tilde{u}^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon \cup \tilde{\Omega}_1^\varepsilon)}^2 \right) + C,
$$

(1.47)

and

$$
\left| \varepsilon^2 a_0 \int_{\tilde{\Omega}_0^\varepsilon} \nabla u^\varepsilon \nabla (g^2 \tilde{u}^\varepsilon) \, dx \right| \leq C \varepsilon \left( \|\nabla (g\tilde{u}^\varepsilon)\|_{L^2(\Omega_1)}^2 + \|g\tilde{u}^\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon \cup \tilde{\Omega}_1^\varepsilon)}^2 \right) + C,
$$

(1.48)
where $C$ does not depend on $\varepsilon$ and $R$.

The proof of this lemma is quite technical and we give it in the next section. We make use of Lemma 1.2.4 and convergence (1.45) to transform identity (1.41)
into the following inequality, valid for small enough \( \varepsilon \):
\[
 a_1 \| \nabla (g\tilde{u}^\varepsilon) \|_{L^2(\Omega^0)}^2 - a_1 \| (\nabla g)\tilde{u}^\varepsilon \|_{L^2(\Omega_i)}^2 - \beta(\lambda_\varepsilon) \| g\tilde{u}^\varepsilon \|_{L^2(\Omega_0^0 \cup \Omega_i)}^2 \\
- 2C\varepsilon \left( \| \nabla (g\tilde{u}^\varepsilon) \|_{L^2(\Omega_i)}^2 + \| g\tilde{u}^\varepsilon \|_{L^2(\Omega_0^0 \cup \Omega_i)}^2 \right) \leq C,
\]
where \( C \) is independent of \( \varepsilon \) and \( R \). Notice that \( \beta(\lambda_\varepsilon) \) is negative and uniformly bounded away from zero as \( \lambda_\varepsilon \to \lambda_0 \). Applying \((1.40)\) to the second term on the left hand side we arrive at
\[
(a_1 - 2C\varepsilon)\| \nabla (g\tilde{u}^\varepsilon) \|_{L^2(\Omega_i)}^2 + \left( -\beta(\lambda_\varepsilon) - \alpha^2 a_1 - 2C\varepsilon \right) \| g\tilde{u}^\varepsilon \|_{L^2(\Omega_0^0 \cup \Omega_i)}^2 \leq C. \tag{1.49}
\]
Hence we should choose \( \alpha \) such that \( -\beta(\lambda_0) - \alpha^2 a_1 \) is positive, i.e.
\[
\alpha < \sqrt{-\beta(\lambda_0)/a_1}.
\]
Since \( g(|x|) \) coincides with \( e^{a|x|} \) on the ball \( B_R \), taking \( \varepsilon \) small enough and restricting the \( L^2 \)-norms to \( B_R \) we arrive at
\[
\| e^{\alpha|x|}\tilde{u}^\varepsilon \|_{L^2(B_R)} \leq C,
\]
where \( C \) does not depend on \( \varepsilon \) and \( R \). Then passing to the limit as \( R \to \infty \) we obtain
\[
\| e^{\alpha|x|}\tilde{u}^\varepsilon \|_{L^2(\mathbb{R}^n)} \leq C. \tag{1.50}
\]

**Step 5.** Despite the fact that the sequence of \( \nabla v^\varepsilon \) is unbounded in \( L^2 \)-norm, the function \( v^\varepsilon \) itself is controlled by \( \tilde{u}^\varepsilon \), see \((1.36)\). Therefore we can get the estimate for the function \( u^\varepsilon \) analogous to \((1.50)\).
\[
\| e^{\alpha|x|}u^\varepsilon \|_{L^2(\mathbb{R}^n)}^2 \leq \| e^{\alpha|x|}\tilde{u}^\varepsilon \|_{L^2(\mathbb{R}^n)}^2 + \sum_{\varepsilon Q_0 \subset \Omega_0^0 \cup \tilde{\Omega}_0^0} \| e^{\alpha|x|}v^\varepsilon \|_{L^2(\varepsilon Q_0)}^2.
\]
In each cell we use inequality \((1.36)\) and
\[
\sup\limits_{x' \in \varepsilon Q} e^{\alpha|x'|} \leq e^{\alpha\sqrt{n}\varepsilon} e^{\alpha|x|}, \quad \forall x \in \varepsilon Q, \tag{1.51}
\]
to obtain
\[
\| e^{\alpha|x|}v^\varepsilon \|_{L^2(\varepsilon Q_0)} \leq C e^{\alpha\sqrt{n}\varepsilon} \| e^{\alpha|x|}\tilde{u}^\varepsilon \|_{L^2(\varepsilon Q_0)} \leq C \| e^{\alpha|x|}\tilde{u}^\varepsilon \|_{L^2(\varepsilon Q_0)}.
\]
and hence, finally,

$$\|e^{\alpha|x|}u^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C$$

uniformly in $\varepsilon$.

**Remark 1.2.5.** It is easy to see that $\nabla \tilde{u}^\varepsilon$ (unlike $\nabla u^\varepsilon$) decays exponentially uniformly in $\varepsilon$ with the same rate as $u^\varepsilon$. Indeed, from (1.51) and (1.27) it follows that

$$\|g\nabla \tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon)}^2 \leq \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon} \sup_{x' \in \varepsilon Q} g\|\nabla \tilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)}^2 \leq C \|g\nabla \tilde{u}^\varepsilon\|_{L^2(\Pi^\varepsilon)}^2,$$

where $\Pi^\varepsilon := \{ \bigcup_{\varepsilon Q} \bigm\varepsilon Q_1 \bigm\text{is such that corresponding} \varepsilon Q_0 \subset \Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon \}$. Since $\|\nabla \tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon)}^2$ and hence $\|g\nabla \tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon)}^2$ are bounded uniformly, we have

$$\|g\nabla \tilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n)}^2 \leq C + C \|g\nabla \tilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon)}^2 = C + C \|\nabla (g\tilde{u}^\varepsilon) - \nabla g\tilde{u}^\varepsilon\|_{L^2(\Omega_1^\varepsilon)}^2 \leq C + C \|\nabla (g\tilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)} + C \alpha \|g\tilde{u}^\varepsilon\|_{L^2(\Omega_1^\varepsilon)}^2.$$

The latter is bounded uniformly in $\varepsilon$ and $R$ due to (1.49) and (1.50). Hence, passing to the limit as $R \to \infty$, we finally arrive at

$$\|e^{\alpha|x|}\nabla \tilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C$$

uniformly in $\varepsilon$.

**Remark 1.2.6.** Estimate (1.24) is sharp in a sense. As we will show later, $u^\varepsilon$ strongly two-scale converges to $u_0$, for which $\sqrt{-\beta(\lambda_0)/a_1}$ is the optimal estimate for its decay exponent, cf. (2.51).

### 1.3 Proof of Lemma [1.2.4]

**Proof.** **Step 1.** First we decompose the function $v^\varepsilon$ in $\Omega_0^\varepsilon$ into the sum of two functions:

$$v^\varepsilon = \tilde{v}^\varepsilon + \hat{v}^\varepsilon,$$

solving the following equations (cf. (1.31)):

$$-a_0 \Delta_y \tilde{v}^\varepsilon(\varepsilon y) - \lambda_\varepsilon \tilde{v}^\varepsilon(\varepsilon y) = \lambda_\varepsilon \langle \tilde{u}^\varepsilon(\varepsilon y) \rangle_y, \ y \in Q_0,$$

$$\tilde{v}^\varepsilon(\varepsilon y) = 0, \ y \in \partial Q_0,$$

1.38
We remind that
\[ -a_0 \Delta_y \hat{v}^\varepsilon(y) = \lambda \hat{v}^\varepsilon(y) = \lambda \langle \hat{u}^\varepsilon(y) - \langle \hat{u}^\varepsilon \rangle_y \rangle, \quad y \in Q_0, \tag{1.55} \]
\[ \hat{v}^\varepsilon(y) = 0, \quad y \in \partial Q_0. \]
The solution of (1.54) could by written in the form
\[ \hat{v}^\varepsilon(y) = \lambda \langle \hat{u}^\varepsilon \rangle_y b_\varepsilon(y), \tag{1.56} \]
where \( b_\varepsilon \) is a solution of (1.12) with \( \lambda = \lambda_\varepsilon \). Due to the uniform (with respect to \( \varepsilon \)) boundedness of the resolvent of the operator \( T \) in the neighbourhood of \( \lambda_0 \), the solution of (1.55) is bounded as follows,
\[ \| \hat{v}^\varepsilon(y) \|_{L^2(Q_0)} = \| (T - \lambda_\varepsilon)^{-1}(\hat{u}^\varepsilon(y) - \langle \hat{u}^\varepsilon \rangle_y) \|_{L^2(Q_0)} \leq \]
\[ \leq C \| \hat{u}^\varepsilon(y) - \langle \hat{u}^\varepsilon \rangle_y \|_{L^2(Q_0)} \leq C \| \nabla_y \hat{u}^\varepsilon \|_{L^2(Q_0)}, \]
here we employed the version of Poincaré inequality for functions from \( H^1(Q_0) \),
\[ \| f - \langle f \rangle_y \|_{L^2(Q_0)} \leq C \| \nabla_y f \|_{L^2(Q_0)}. \]
After the rescaling we obtain that \( \hat{v}^\varepsilon \) is relatively small compared to \( \nabla \hat{u}^\varepsilon \),
\[ \| \hat{v}^\varepsilon(x) \|_{L^2(\varepsilon Q_0)} \leq \varepsilon C \| \nabla \hat{u}^\varepsilon(x) \|_{L^2(\varepsilon Q)}, \tag{1.57} \]
where \( C \) in the inequality does not depend on \( \varepsilon \) or \( \xi \in \mathbb{Z}^n \).

**Step 2.** At this stage we will need several inequalities which follow from the properties of \( g \) and \( \hat{u}^\varepsilon \).

**Proposition 1.3.1.** The following estimates are valid for small enough \( \varepsilon \) with constants independent of \( \varepsilon \) and the choice of particular \( \varepsilon Q \):
\[ \| g^2 \hat{u}^\varepsilon \|_{L^2(\varepsilon Q)} \| \nabla \hat{u}^\varepsilon \|_{L^2(\varepsilon Q)} \leq C \left( \| \nabla (g \hat{u}^\varepsilon) \|_{L^2(\varepsilon Q)}^2 + \| g \hat{u}^\varepsilon \|_{L^2(\varepsilon Q)}^2 \right), \tag{1.58} \]
\[ \| \nabla \hat{u}^\varepsilon \|_{L^2(\varepsilon Q)} \| \nabla (g^2 \hat{u}^\varepsilon) \|_{L^2(\varepsilon Q)} \leq C \left( \| \nabla (g \hat{u}^\varepsilon) \|_{L^2(\varepsilon Q)}^2 + \| g \hat{u}^\varepsilon \|_{L^2(\varepsilon Q)}^2 \right), \tag{1.59} \]
\[ \| \nabla \hat{u}^\varepsilon \|_{L^2(\varepsilon Q)} \| \nabla (g^2 \hat{u}^\varepsilon) \|_{L^2(\varepsilon Q)} \leq C \left( \| \nabla (g \hat{u}^\varepsilon) \|_{L^2(\varepsilon Q)}^2 + \| g \hat{u}^\varepsilon \|_{L^2(\varepsilon Q)}^2 \right). \tag{1.60} \]

**Proof.** We remind that
\[ \sup_{\varepsilon Q} g \leq e^{a \sqrt{\varepsilon}} g(x) \leq C g(x), \quad x \in \varepsilon Q, \tag{1.61} \]
for small enough \( \varepsilon \). We apply (1.27), (1.40) and (1.61) to get (1.58):

\[
\| g^2 \tilde{u}^\varepsilon \|_{L^2(\Omega)} \| \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \leq C \left\| \left( \sup_{\varepsilon Q} g \right) g \tilde{u}^\varepsilon \right\|_{L^2(\varepsilon Q)} \| \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q_1)} \leq C \| g \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \left( \| \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} + \| \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \right) = C \| g \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \left( \| \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \| g \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \right) \leq C \left( \| \nabla (g \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q)}^2 + \| g \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)}^2 \right).
\]

The last inequality in the chain follows from the elementary

\[
|ab| \leq \frac{1}{2} (a^2 + b^2).
\]

The proof of (1.59) and (1.60) is analogous:

\[
\| \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \| \nabla (g^2 \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q)} = \| \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \| \nabla (g^2 \tilde{u}^\varepsilon) + g^2 \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \leq C \left( \| g^2 \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \| \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} + \| g \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q_1)} \right) \leq C \left( \| \nabla (g \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q)}^2 + \| g \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)}^2 \right).
\]

\[
\| \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \| \nabla (g^2 \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q)} = \| \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \| \nabla (g^2 \tilde{u}^\varepsilon) + g^2 \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \leq C \left( \| g^2 \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \| \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} + \left( \left( \sup_{\varepsilon Q} g \right) \nabla \tilde{u}^\varepsilon \right) \right)^2 \leq C \left( \| \nabla (g \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q)}^2 + \| g \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)}^2 \right).
\]

Substituting (1.53) into the second term on the left hand side of (1.47) we obtain

\[
\lambda_\varepsilon \int_{\Omega_0} g^2 v^\varepsilon \tilde{u}^\varepsilon \, dx = \lambda_\varepsilon \int_{\Omega_0} g^2 \tilde{v}^\varepsilon \tilde{u}^\varepsilon \, dx + \lambda_\varepsilon \int_{\Omega_0} g^2 \tilde{v}^\varepsilon \tilde{u}^\varepsilon \, dx.
\]

Let us show that \( \lambda_\varepsilon \int_{\Omega_0} g^2 \tilde{v}^\varepsilon \tilde{u}^\varepsilon \, dx \) is relatively small. Indeed, applying inequalities
(1.57) and (1.58) in each cell we obtain

\[ \lambda \varepsilon \int_{\Omega_0^\varepsilon} g^2 \tilde{\tilde{u}}^\varepsilon \tilde{v}^\varepsilon \, dx \leq \lambda \varepsilon \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \| g^2 \tilde{\tilde{u}}^\varepsilon \|_{L^2(\varepsilon Q_0)} \| \tilde{v}^\varepsilon \|_{L^2(\varepsilon Q_0)} \leq \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon C \left( \| \nabla (g \tilde{\tilde{u}}^\varepsilon) \|_{L^2(\varepsilon Q_1)}^2 + \| g \tilde{\tilde{u}}^\varepsilon \|_{L^2(\varepsilon Q)}^2 \right). \] (1.62)

Considering sets

\[ \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q \quad \text{and} \quad \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q_1, \]

one can notice that they are “nearly” equal to

\[ \Omega_0^\varepsilon \cup \Omega_1^\varepsilon \quad \text{and} \quad \Omega_1^\varepsilon, \]

respectively. Namely,

\[ \Omega_0^\varepsilon \cup \Omega_1^\varepsilon = \left( \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q \right) \cup \Omega_{1,+}^\varepsilon \setminus \Omega_{1,-}^\varepsilon, \]
\[ \Omega_1^\varepsilon = \left( \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q_1 \right) \cup \Omega_{1,+}^\varepsilon \setminus \Omega_{1,-}^\varepsilon, \] (1.63)

where

\[ \Omega_{1,-}^\varepsilon = \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q_0 \cap \Omega_2; \]
\[ \Omega_{1,+}^\varepsilon = \bigcup_{\varepsilon Q_0 \cap \Omega_2 \neq \emptyset} \varepsilon Q_0 \cap \Omega_1^\varepsilon. \]

We introduce two ‘correctors’

\[ r^\varepsilon = \| \nabla (g \tilde{\tilde{u}}^\varepsilon) \|_{L^2(\Omega_{1,-}^\varepsilon)}^2 + \| g \tilde{\tilde{u}}^\varepsilon \|_{L^2(\Omega_{1,-}^\varepsilon)}^2, \] (1.64)

and

\[ r_1^\varepsilon = \| g \tilde{\tilde{u}}^\varepsilon \|_{L^2(\Omega_{1,+}^\varepsilon \cup \Omega_{1,-}^\varepsilon)}^2. \]

Then inequality (1.62) transforms into

\[ \int_{\Omega_0^\varepsilon} g^2 \tilde{\tilde{u}}^\varepsilon \tilde{v}^\varepsilon \, dx \leq \varepsilon C \left( \| \nabla (g \tilde{\tilde{u}}^\varepsilon) \|_{L^2(\Omega_{1}^\varepsilon)}^2 + \| g \tilde{\tilde{u}}^\varepsilon \|_{L^2(\Omega_{1,+} \cup \Omega_{1,-}^\varepsilon)}^2 + r^\varepsilon \right). \] (1.65)
Step 3. Now we consider the term $\lambda_\varepsilon \int_{\Omega_0} g^2 \tilde{v} \tilde{u} \, dx$. We substitute (1.56) to obtain
\[
\lambda_\varepsilon \int_{\Omega_0} g^2 \tilde{v} \tilde{u} \, dx = \lambda_\varepsilon \varepsilon^n \sum_{eQ_0 \subset \Omega_0} \int_Q g^2 \tilde{u} (\varepsilon y)b_\varepsilon(y) (\tilde{u} \varepsilon y) \, dy,
\]
where $b_\varepsilon$ is considered as a periodic function on $\mathbb{R}^n$, $b_\varepsilon(y + \xi) = b_\varepsilon(y)$, $\xi \in \mathbb{Z}^n$, and $(\tilde{u} \varepsilon y)_Q = (\tilde{u} \varepsilon y)_Q(y) = \int_{Q \cap \Omega} \tilde{u} \varepsilon (\varepsilon y') dy'$ is a step function that takes constant values on each cell $Q$. Notice also that $\beta(\lambda_\varepsilon) - \lambda_\varepsilon = \lambda_\varepsilon^2 (b_\varepsilon)_Q$. Then, keeping in mind (1.63), we obtain
\[
\Lambda_\varepsilon := \left| \lambda_\varepsilon \int_{\Omega_0} g^2 \tilde{v} \tilde{u} \, dx - (\beta(\lambda_\varepsilon) - \lambda_\varepsilon) \int_{\Omega_0} \int_{\Omega_0} g^2 (\tilde{u} \varepsilon)^2 \, dx \right| \leq
\]
\[
\leq C \varepsilon^n \sum_{eQ_0 \subset \Omega_0} \left| \int_Q g^2 \tilde{u} (\varepsilon y)b_\varepsilon(y) (\tilde{u} \varepsilon y) \, dy - (b_\varepsilon)_y \int_Q g^2 (\varepsilon y) (\tilde{u} \varepsilon (\varepsilon y))^2 \, dy \right| +
\]
\[
+ C r_1 \leq C \varepsilon^n \sum_{eQ_0 \subset \Omega_0} \left| \langle \tilde{u} \varepsilon y \rangle_Q \int_Q (g^2 \tilde{u} \varepsilon - (g^2 \tilde{u} \varepsilon)_y) b_\varepsilon \, dy \right| +
\]
\[
+ C \varepsilon^n \sum_{eQ_0 \subset \Omega_0} \left| (b_\varepsilon)_y \int_Q (g^2 \tilde{u} \varepsilon - (g^2 \tilde{u} \varepsilon)_y) \tilde{u} \varepsilon \, dy \right| + C r_1 \varepsilon.
\]
We will separately estimate terms contained in the last expression. The mean value of $\tilde{u} \varepsilon$ is bounded by its norm in $L^2$ by Hölder inequality
\[
\langle \tilde{u} \varepsilon (\varepsilon y) \rangle_Q^2 = \left[ \int_Q \tilde{u} \varepsilon \, dy \right]^2 \leq \int_Q (\tilde{u} \varepsilon)^2 \, dy \int_Q 1 \, dy = \| \tilde{u} \varepsilon (\varepsilon y) \|_{L^2(Q)}^2.
\]
Similarly,
\[
\langle b_\varepsilon \rangle_y \leq \| b_\varepsilon \|_{L^2(Q_0)} \leq C,
\]
where $C$ does not depend on $\varepsilon$ due to the uniform boundedness of the resolvent.
\((T - \lambda)^{-1}\) in the neighbourhood of \(\lambda_0\). Via the Poincaré inequality we derive

\[
\left| \int_Q \left( g^2 \tilde{u} - \langle g^2 \tilde{u} \rangle_y \right) \tilde{u} \, dy \right| \leq \| g^2 \tilde{u} - \langle g^2 \tilde{u} \rangle_y \|_{L^2(Q)} \| \tilde{u} \|_{L^2(Q)} \leq \\
\leq C \| \nabla_y (g^2 \tilde{u}) \|_{L^2(Q)} \| \tilde{u} \|_{L^2(Q)},
\]

and, similarly,

\[
\left| \int_Q \left( g^2 \tilde{u} - \langle g^2 \tilde{u} \rangle_y \right) b_y \, dy \right| \leq C \| \nabla_y (g^2 \tilde{u}) \|_{L^2(Q)},
\]

(1.68)

with constants independent of \(\varepsilon\) and \(\xi\) (see (1.1)). Applying inequalities (1.67)–(1.68) and then (1.59) to (1.66) we arrive at

\[
\Lambda_\varepsilon \leq \varepsilon^n \sum_{\varepsilon Q_0 \subset \Omega_0} \| \tilde{u}^\varepsilon(\varepsilon y) \|_{L^2(Q)} \| \nabla_y (g^2(\varepsilon y) \tilde{u}^\varepsilon(\varepsilon y)) \|_{L^2(Q)} + C r_1^\varepsilon \leq \\
\leq \varepsilon C \sum_{\varepsilon Q_0 \subset \Omega_0} \| \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)} \| \nabla (g^2 \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q)} + C r_1^\varepsilon \leq \\
\leq \varepsilon C \sum_{\varepsilon Q_0 \subset \Omega_0} \left( \| \nabla (g \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q_1)}^2 + \| g \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)}^2 \right) + C r_1^\varepsilon \leq \\
\leq \varepsilon C \left( \| \nabla (g \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q_1)}^2 + \| g \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q)}^2 \right) + C r_1^\varepsilon,
\]

(1.69)

where \(C\) is \(\varepsilon\)-independent. At the last step we used formulas (1.63) and (1.64). Since the correctors \(r^\varepsilon, r_1^\varepsilon\) are uniformly bounded, inequalities (1.65) and (1.69) together imply the validity of (1.47).

**Step 4.** Finally, it is not difficult to obtain similarly (1.48) via (1.36), (1.59)
\[ \varepsilon^2 a_0 \int_{\Omega_0} \nabla u^\varepsilon \nabla (g^2 \tilde{u}^\varepsilon) \, dx \leq \varepsilon^2 C \sum_{\varepsilon Q_0 \subset \Omega_0} \| \nabla u^\varepsilon \|_{L^2(\varepsilon Q_0)} \| \nabla (g^2 \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q_0)} \leq \]
\[ \leq \varepsilon' \sum_{\varepsilon Q_0 \subset \Omega_0} \left( \| \varepsilon \nabla v^\varepsilon \|_{L^2(\varepsilon Q_0)} + \varepsilon \| \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q_0)} \right) \| \nabla (g^2 \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q_0)} \leq \]
\[ \leq \varepsilon' \sum_{\varepsilon Q_0 \subset \Omega_0} \left( \| \nabla (g \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q_0)}^2 + \| \nabla \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q_0 \cup \Omega_1^\varepsilon)}^2 \right) \leq \]
\[ \leq \varepsilon' \left( \| \nabla (g \tilde{u}^\varepsilon) \|_{L^2(\varepsilon Q_1^a)}^2 + \| g \tilde{u}^\varepsilon \|_{L^2(\varepsilon Q_0 \cup \Omega_1^\varepsilon)}^2 \right) \]

for small enough \( \varepsilon \).

Notice that all the estimates obtained in this section are independent of \( R \). \( \Box \)
Chapter 2

Two-scale convergence of eigenfunctions and convergence of spectra

In this chapter we study the convergence properties of the localised eigenfunctions of the operator $A_\varepsilon$ and convergence of its spectrum. We list the definitions and some properties of the two-scale convergence, see [2,38,48,49], in the first section. We also formulate several auxiliary statements (analogous to those in [49]) which are necessary for obtaining the two-scale convergence of the eigenfunctions of $A_\varepsilon$ and for the derivation of the limit equation. In Section 2.2 we prove, relying on the uniform exponential decay, the main results on the two-scale convergence of the eigenfunctions and the subsequent convergence of the point spectrum of $A_\varepsilon$.

In Section 2.3 we provide a proof of stability of the essential spectrum of the two-scale homogenised operator with respect to the compact perturbation of its coefficients, thereby establishing the Hausdorff convergence of the spectra of $A_\varepsilon$ to the spectrum of the homogenised operator $A_0$.

2.1 Some properties of two-scale convergence

Let $\Omega$ be an arbitrary region in $\mathbb{R}^n$, in particular $\Omega = \mathbb{R}^n$. Denote by $\Box$ the unit cube $[0,1)^n$. We consider all functions of the form $u(x,y)$ to be 1-periodic in $y$ in each coordinate.

Definition 2.1.1. We say that a bounded in $L^2(\Omega)$ sequence $v_\varepsilon$ is weakly two-
scale convergent to a function $v \in L^2(\Omega \times \Box)$, $v_\varepsilon(x) \overset{\varepsilon \to 0}{\rightharpoonup} v(x,y)$, if
\[
\lim_{\varepsilon \to 0} \int_{\Omega} v_\varepsilon(x) \varphi(x)b\left(\frac{x}{\varepsilon}\right) \, dx = \int_{\Omega} \int_{\Box} v(x,y) \varphi(x)b(y) \, dy \, dx
\]
for all $\varphi \in C^\infty_0(\Omega)$ and all $b \in C^\infty_{\text{per}}(\Box)$ (where $C^\infty_{\text{per}}(\Box)$ is the set of 1-periodic functions from $C^\infty(\mathbb{R}^n)$).

**Definition 2.1.2.** We say that a bounded in $L^2(\Omega)$ sequence $u_\varepsilon$ is strongly two-scale convergent to a function $u \in L^2(\Omega \times \Box)$, $u_\varepsilon(x) \overset{\varepsilon \to 0}{\rightharpoonup} u(x,y)$, if
\[
\lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon(x) v_\varepsilon(x) \, dx = \int_{\Omega} \int_{\Box} u(x,y) v(x,y) \, dy \, dx
\]
for all $v_\varepsilon(x) \overset{\varepsilon \to 0}{\rightharpoonup} v(x,y)$.

**Proposition 2.1.3.** (Properties of the two-scale convergence.)
(i) If $u_\varepsilon(x) \overset{\varepsilon \to 0}{\rightharpoonup} u(x,y)$ and $a \in L^\infty_{\text{per}}(\Box)$ then
\[
a(x/\varepsilon) u_\varepsilon(x) \overset{\varepsilon \to 0}{\rightharpoonup} a(y) u(x,y).
\]
(ii) $v_\varepsilon(x) \overset{\varepsilon \to 0}{\rightharpoonup} v(x,y)$ if and only if $v_\varepsilon(x) \overset{\varepsilon \to 0}{\rightharpoonup} v(x,y)$ and
\[
\lim_{\varepsilon \to 0} \int_{\Omega} v_\varepsilon^2 \, dx = \int_{\Omega} \int_{\Box} v^2 \, dy \, dx.
\]
(iii) If $f_\varepsilon(x) \to f(x)$ in $L^2(\Omega)$, then $f_\varepsilon(x) \overset{\varepsilon \to 0}{\rightharpoonup} f(x)$.
(iv) A sequence $u_\varepsilon$ bounded in $L^2(\Omega)$ is compact in the sense of weak two-scale convergence.

**Proposition 2.1.4.** (The mean value property of periodic functions.) Let $\Phi(y) \in L^1_{\text{per}}(\Box)$. Then for each $\phi(x) \in C^\infty_0(\mathbb{R}^n)$ we have
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \phi(x) \Phi(x/\varepsilon) \, dx = \langle \Phi \rangle_y \int_{\mathbb{R}^n} \phi(x) \, dx.
\]

The potential vector space $V_{\text{pot}}$ is defined as a closure of the set $\{\nabla \varphi : \varphi \in C^\infty_{\text{per}}(\Box)\}$ in $L^2(\Box)^n$. We say that a vector $b \in L^2(\Box)^n$ is solenoidal ($b \in V_{\text{solenoidal}}$)
if it is orthogonal to all potential vectors. Thus,

\[ L^2(\square)^n = V_{\text{pot}} \oplus V_{\text{sol}}, \]

and

\[ L^2(\Omega \times \square)^n = L^2(\Omega, V_{\text{pot}}) \oplus L^2(\Omega, V_{\text{sol}}). \]

**Lemma 2.1.5.** Let \( u_\varepsilon \) and \( \varepsilon \nabla u_\varepsilon \) be bounded in \( L^2(\mathbb{R}^n) \). Then (up to a subsequence)

\[ u_\varepsilon(x) \xrightarrow{2} u(x, y) \in L^2(\mathbb{R}^n, H^1_{\text{per}}), \]

\[ \varepsilon \nabla u_\varepsilon(x) \xrightarrow{2} \nabla_y u(x, y), \]

where \( H^1_{\text{per}} = H^1_{\text{per}}(\square) \) is the Sobolev space of periodic functions.

**Lemma 2.1.6.** Let \( u_\varepsilon \in H^1(\mathbb{R}^n) \),

\[ u_\varepsilon(x) \xrightarrow{2} u(x) \in H^1(\mathbb{R}^n), \tag{2.1} \]

and \( \nabla u_\varepsilon \) is bounded in \( L^2(\mathbb{R}^n) \). Then, up to a subsequence,

\[ \nabla u_\varepsilon(x) \xrightarrow{2} \nabla u(x) + v(x, y), \text{ where } v \in L^2(\mathbb{R}^n, V_{\text{pot}}). \tag{2.2} \]

**Lemma 2.1.7.** Let (2.1) and (2.2) be valid. Let also

\[ \lim_{\varepsilon \to 0} \int_{\Omega_1} a_1 \nabla u_\varepsilon(x) \cdot \nabla_y w(\varepsilon^{-1}x) \varphi(x) \, dx = 0 \tag{2.3} \]

for any \( \varphi \in C^\infty_0(\Omega_1) \) and \( w \in C^\infty_{\text{per}}(\square) \). Then the following weak convergence of the flows takes place:

\[ a_1 \Theta_{Q_1}(\varepsilon^{-1}x) \nabla u_\varepsilon(x) \rightharpoonup A^\text{hom} \nabla u(x) \text{ in } \Omega_1, \]

where homogenised matrix \( A^\text{hom} \) is defined by (1.10).

The proofs of the listed statements repeat the proofs of the corresponding assertions in [48] with no or only small alterations, and are not given here. The following is an important definition of the strong two-scale resolvent convergence of operators.

**Definition 2.1.8.** Let \( A_\varepsilon, \varepsilon > 0 \), and \( A_0 \) be non-negative self-adjoint operators in \( L^2(\mathbb{R}^n) \) and \( \mathcal{H}_0 \subset L^2(\mathbb{R}^n \times Q) \), see (1.13), respectively. We say that \( A_\varepsilon \xrightarrow{2} \)
in the sense of the strong two-scale resolvent convergence if for any $\lambda > 0$
\((A_\varepsilon + \lambda I)^{-1} f_\varepsilon \xrightarrow{\varepsilon \to 0} (A_0 + \lambda I)^{-1} f_0\) as long as \(f_\varepsilon \xrightarrow{\varepsilon \to 0} f_0\).

### 2.2 Strong two-scale convergence of the eigenfunctions and multiplicity of the eigenvalues of \(A_\varepsilon\)

In this section we will show that the normalised eigenfunctions \(u_\varepsilon\) are compact in the sense of strong two-scale convergence. Namely, provided \(\lambda_\varepsilon \to \lambda_0\), a sequence of normalised eigenfunctions \(u_\varepsilon\) of the operator \(A_\varepsilon\) strongly two-scale converges, up to a subsequence, to a function \(u^0(x,y)\). Using the properties of two-scale convergence we then pass to a limit in integral identity (1.6) with a specially chosen test function. As a result we obtain in the limit integral identity (1.18) which implies that \(\lambda_0\) and \(u^0(x,y)\) are an eigenvalue and a corresponding eigenfunction of \(A_0\). This, together with the results of [29], allows us to establish an ‘asymptotic one-to-one correspondence’ between isolated eigenvalues and corresponding eigenfunctions of the operators \(A_\varepsilon\) and \(A_0\).

**Theorem 2.2.1.** Under the assumptions of Theorem 1.2.2 \(\lambda_0\) is an eigenvalue of the operator \(A_0\). Moreover, there exists a subsequence \(\varepsilon\) such that eigenfunctions \(u_\varepsilon\) of the operator \(A_\varepsilon\) strongly two-scale converge to an eigenfunction \(u^0(x,y)\) of \(A_0\) corresponding to the eigenvalue \(\lambda_0\).

**Proof.** Step 1. In order to establish strong two-scale convergence of the eigenfunctions \(u_\varepsilon = \tilde{u}_\varepsilon + v_\varepsilon\) we prove it for each of its components separately. The gradient of \(\tilde{u}_\varepsilon\) is bounded in \(L^2\)-norm uniformly in \(\varepsilon\). Naively speaking, this means that \(\tilde{u}_\varepsilon\) itself is a function of slow variation and its two-scale limit should not depend on the fast variable \(y\). Then one can expect that the sequence \(\tilde{u}_\varepsilon\) is compact in a usual \(L^2\)-norm sense.

From (1.50) it follows that

\[
\|e^{\alpha R} \tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^n \setminus B_R)} \leq \|e^{\alpha |x|} \tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^n \setminus B_R)} \leq C.
\]

Then

\[
\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^n \setminus B_R)} \leq C e^{-\alpha R} \tag{2.4}
\]

with \(C\) independent of \(\varepsilon\) and \(R\). From this one can easily conclude that \(\tilde{u}_\varepsilon\) is
weakly compact in $H^1(\mathbb{R}^n)$ and strongly compact in $L^2(\mathbb{R}^n)$. Indeed, since $\tilde{u}^\varepsilon$ is bounded in $H^1(\mathbb{R}^n)$ uniformly in $\varepsilon$ (see (1.28)),

$$\tilde{u}^\varepsilon(x) \rightharpoonup u_0(x) \text{ in } H^1(\mathbb{R}^n),$$

(2.5)

up to a subsequence due to the weak compactness of a bounded set in $H^1(\mathbb{R}^n)$. It is well known that $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$ when $\Omega$ is bounded. Hence, for any fixed $R$ function $\tilde{u}^\varepsilon$ converges to $u_0$ weakly in $H^1(B_R)$ and strongly in $L^2(B_R)$ up to a subsequence. Considering a sequence of balls $B_R, R \in \mathbb{N}$, one can use the method of extracting a diagonal subsequence to obtain a sequence converging in any ball $B_R$,

$$\tilde{u}^\varepsilon \to u_0 \text{ in } L^2(B_R)$$

(2.6)

for any $R > 0$.

For any $\delta > 0$ we can choose $R$ such that $\|u_0\|_{L^2(\mathbb{R}^n \setminus B_R)}^2 < \delta/3$ and $\|	ilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n \setminus B_R)}^2 < \delta/3$ for all sufficiently small $\varepsilon$ (the latter follows from (2.4)). From (2.6) it follows that $\|u_0 - \tilde{u}^\varepsilon\|_{L^2(B_R)}^2 < \delta/3$ for sufficiently small $\varepsilon$. So we conclude that

$$\|u_0 - \tilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n)}^2 \leq \|u_0 - \tilde{u}^\varepsilon\|_{L^2(B_R)}^2 + \|u_0\|_{L^2(\mathbb{R}^n \setminus B_R)}^2 + \|	ilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n \setminus B_R)}^2 < \delta$$

for small enough $\varepsilon$. Hence, up to a subsequence, we have proved convergence

$$\tilde{u}^\varepsilon \to u_0 \text{ in } L^2(\mathbb{R}^n).$$

(2.7)

Then from the properties of two-scale convergence (Proposition 2.1.3 (iii)) we conclude that

$$\tilde{u}^\varepsilon(x) \overset{2}{\to} u_0(x).$$

(2.8)

**Step 2.** Now let us consider $v^\varepsilon$. Formulas (1.53) and (1.56) show that $v^\varepsilon$ is of a two-scale nature. One can expect that its two-scale limit depends both on $x$ and $y$. Since the coefficient $a(x,\varepsilon)$ on $\tilde{\Omega}_0^\varepsilon$ is defined very loosely we consider the behaviour of $v^\varepsilon$ on $\tilde{\Omega}_0^\varepsilon$ separately. We denote by $v^\varepsilon_1$ and $v^\varepsilon_2$ the restrictions $v^\varepsilon|_{\Omega_0^\varepsilon}$ and $v^\varepsilon|_{\tilde{\Omega}_0^\varepsilon}$ respectively, extending them by zero to the rest of $\mathbb{R}^n$.

**Lemma 2.2.2.** The following convergence properties are valid for $v^\varepsilon_1$ (up to a
subsequence): 

\[ v_1^\varepsilon(x) \xrightarrow{2} v(x, y) \in L^2(\Omega_1, H^1_0(Q_0)), \]

\[ \varepsilon \nabla v_1^\varepsilon(x) \xrightarrow{2} \nabla_y v(x, y), \]

where \( v(x, y) \) is a solution to the following problem:

\[ -a_0 \Delta_y v - \lambda_0 v = \lambda_0 u_0, \quad y \in Q_0. \tag{2.9} \]

Here \( u_0 \) is a function from (2.8).

**Proof.** The function \( v_1^\varepsilon \in H^1(\Omega_0^\varepsilon) \) satisfies the following differential equation:

\[ -\varepsilon^2 a_0 \Delta v_1^\varepsilon - \lambda_0 v_1^\varepsilon = \lambda_0 \tilde{u}_\varepsilon, \quad x \in \Omega_0^\varepsilon. \tag{2.10} \]

Let us rewrite it in the form

\[ -\varepsilon^2 a_0 \Delta v_1^\varepsilon - \lambda_0 v_1^\varepsilon = \lambda_0 \tilde{u}_\varepsilon \left( \Theta_{Q_0}(x/\varepsilon) - \Theta_{\tilde{Q}_0}(x) \right), \quad x \in \Omega_1. \tag{2.11} \]

We understand the term \( \Theta_{Q_0}(y) \) as a characteristic function of \( Q_0 \) in \( Q \) extended by periodicity on \( \mathbb{R}^n \). Since \( \tilde{u}_\varepsilon \) is bounded in \( L^2(\mathbb{R}^n) \) and the Lebesgue measure of \( \tilde{Q}_0^\varepsilon \) tends to zero, we have

\[ \| \tilde{u}_\varepsilon \|_{L^2(\mathbb{R}^n)}^2 \leq \| \tilde{u}_\varepsilon \|_{L^2(\mathbb{R}^n)} \| \Theta_{\tilde{Q}_0} \|_{L^2(\mathbb{R}^n)} \to 0. \]

By Proposition 2.1.3 (i) for any \( f_\varepsilon(x) \xrightarrow{2} f(x, y) \) it is true that \( f_\varepsilon(x) \Theta_{Q_0}(x/\varepsilon) \xrightarrow{2} f(x, y) \Theta_{Q_0}(y) \). Since \( \tilde{u}_\varepsilon \) strongly two-scale converges to \( u_0 \), by the above and the definition of the strong two-scale convergence we have

\[ \int_{\mathbb{R}^n} \tilde{u}_\varepsilon(x) f_\varepsilon(x) \Theta_{Q_0}(x/\varepsilon) dx \to \int_{\mathbb{R}^n} \int_{Q} u_0(x) f(x, y) \Theta_{Q_0}(y) dx \]

with arbitrary \( f_\varepsilon(x) \xrightarrow{2} f(x, y) \). But this implies \( \tilde{u}_\varepsilon(x) \Theta_{Q_0}(x/\varepsilon) \xrightarrow{2} u_0(x) \Theta_{Q_0}(y) \).

Hence we conclude that

\[ \lambda_0 \tilde{u}_\varepsilon \left( \Theta_{Q_0}(x/\varepsilon) - \Theta_{\tilde{Q}_0}(x) \right) \xrightarrow{2} \lambda_0 \Theta_{Q_0}(y) u_0(x) \in L^2(\Omega_1 \times \Box). \tag{2.12} \]

Following [48] we consider the more general problem

\[ z_\varepsilon \in H^1(\Omega_0^\varepsilon), \quad -\varepsilon^2 a_0 \Delta z_\varepsilon - \lambda_0 z_\varepsilon = f_\varepsilon, \quad f_\varepsilon \in L^2(\Omega_0^\varepsilon). \tag{2.13} \]
(It is implicit that $f_\varepsilon = z_\varepsilon = 0$ in $\mathbb{R}^n \setminus \Omega_0^\varepsilon$.)

**Proposition 2.2.3.** Let

$$f^\varepsilon(x) \overset{2}{\rightharpoonup} f(x, y).$$

Then

$$z^\varepsilon(x) \overset{2}{\rightharpoonup} z(x, y) \in L^2(\Omega_1, H^1_0(Q_0)),$$

$$\varepsilon \nabla z^\varepsilon(x) \overset{2}{\rightharpoonup} \nabla_y z(x, y),$$

where function $z(x, y)$ solves the following equation:

$$- a_0 \Delta_y z - \lambda_0 z = f, \quad y \in Q_0.$$  \hspace{1cm} (2.14)

**Proof.** One can easily derive an estimate for $z^\varepsilon$ analogous to (1.36), applying to (2.13) a reasoning similar to those for the solution of equation (1.30):

$$a_0 \varepsilon^2 \|\nabla z^\varepsilon(x)\|_{L^2(Q_0^\varepsilon)} + \|z^\varepsilon(x)\|_{L^2(Q_0^\varepsilon)} \leq C \|f^\varepsilon(x)\|_{L^2(Q_0^\varepsilon)},$$

with $C$ independent of $\varepsilon$. Since $f^\varepsilon$ weakly two-scale converges, it is bounded. Then $z^\varepsilon$ and $\varepsilon \nabla z^\varepsilon$ are also bounded, and we can apply Lemma 2.1.5:

$$z^\varepsilon(x) \overset{2}{\rightharpoonup} z(x, y) \in L^2(\Omega_1, H^1_{\text{per}}),$$

$$\varepsilon \nabla z^\varepsilon(x) \overset{2}{\rightharpoonup} \nabla_y z(x, y).$$

Equation (2.14) follows by a straightforward passing to the limit in the integral identity corresponding to (2.13) with appropriately chosen test functions. The full proof can be found in [48] and applies to the present situation with no alteration.

**Remark 2.2.4.** Validity of $z \in L^2(\Omega_1, H^1_0(Q_0))$, i.e. that $z$ vanishes on the boundary of $Q_0$, follows from $z \in L^2(\Omega_1, H^1_{\text{per}})$ and the obvious convergence property

$$0 \equiv z^\varepsilon(x) \Theta_{Q_1}(x/\varepsilon) \overset{2}{\rightharpoonup} z(x, y) \Theta_{Q_1}(y).$$

\[ \square \]

The above proposition together with (2.12) establishes a “weak” form of the statement of the lemma, i.e. weak two-scale convergence of $v^\varepsilon_1$ to the solution of (2.9). We now prove that the convergence is actually strong, following again [48]. Multiply (2.10) and (2.13) by $z^\varepsilon$ and $v^\varepsilon_1$ respectively and integrate by parts. The left hand sides of the resulting equalities are identical. So, equating the right
hand sides, we obtain the following identity
\[
\int_{\Omega_1} f^\varepsilon v_1^\varepsilon \, dx = \lambda_\varepsilon \int_{\Omega_1} \tilde{u}^\varepsilon z^\varepsilon \, dx. \tag{2.15}
\]
Since \( \tilde{u}^\varepsilon \) strongly two-scale converges, then by the definition we have
\[
\lim_{\varepsilon \to 0} \lambda_\varepsilon \int_{\Omega_1} \tilde{u}^\varepsilon z^\varepsilon \, dx = \lambda_0 \int_{\Omega_1} \int_{Q_0} u_0(x)z(x,y) \, dy \, dx.
\]
Multiplying (2.9) and (2.14) by \( z \) and \( v \) respectively and integrating by parts it is easy to see that
\[
\lambda_0 \int_{\Omega_1} \int_{Q_0} u_0(x)z(x,y) \, dy \, dx = \int_{\Omega_1} \int_{Q_0} f(x,y)v(x,y) \, dy \, dx. \tag{2.16}
\]
Since the right hand side of (2.15) converges to the left hand side of (2.16) we conclude that:
\[
\lim_{\varepsilon \to 0} \int_{\Omega_1} f^\varepsilon v_1^\varepsilon \, dx = \int_{\Omega_1} \int_{Q_0} f(x,y)v(x,y) \, dy \, dx
\]
for any weakly two-scale convergent sequence \( f^\varepsilon \). Hence, by the definition of the strong two-scale convergence,
\[
v_1^\varepsilon(x) \to v(x,y).
\]
\[ \square \]

**Lemma 2.2.5.** The sequence of functions \( v_2^\varepsilon \) converges to zero in the sense of strong two-scale convergence:
\[
v_2^\varepsilon \to 0 \text{ as } \varepsilon \to 0.
\]

**Proof.** Straightforward from (1.46) and Proposition 2.1.3 (iii). \[ \square \]

Combining (2.8) with Lemmas 2.2.2 and 2.2.5 we arrive at
\[
u^\varepsilon(x) \to u^0(x,y) = u_0(x) + v(x,y), \tag{2.17}
\]
where \( u_0 \in H^1(\mathbb{R}^n), \ v \in L^2(\Omega_1, H^1_0(Q_0)) \).
Step 3. Now it remains to show that $\lambda_0$ and $u^0(x,y)$ are an eigenvalue and the corresponding eigenfunction of the limit operator $A_0$, i.e. that $u^0(x,y)$ satisfies (1.19). In order to do that we need to choose an appropriate test-function $\psi^\varepsilon$ and pass to the limit in the integral identity

$$
\varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx + a_1 \int_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx + \int_{\bar{\Omega}_0^\varepsilon} \tilde{a}_0 \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx + a_2 \int_{\Omega_2^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx = \lambda_0 \int_{\mathbb{R}^n} u^\varepsilon \psi^\varepsilon \, dx
$$

(2.18)
corresponding to the original eigenvalue problem (1.2)–(1.3). Let us take

$$
\psi^\varepsilon(x) = \psi_0(x) + \varphi(x) b(\varepsilon^{-1} x),
$$

$$
\psi_0 \in C_0^\infty(\mathbb{R}^n), \varphi \in C_0^\infty(\Omega_1), b(y) \in C_0^\infty(Q_0),
$$

(2.19)
and consider each term of (2.18) separately. Let us expand the first term:

$$
\varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla \psi^\varepsilon \, dx = \varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla \tilde{u}^\varepsilon \nabla \psi^\varepsilon \, dx +
$$

$$
+ \varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla u^\varepsilon \left( \nabla \psi_0 + b(\varepsilon^{-1} x) \nabla \varphi \right) \, dx + a_0 \int_{\Omega_0^\varepsilon} \varepsilon \nabla u^\varepsilon \nabla_y b(\varepsilon^{-1} x) \, dx.
$$

Since $\nabla \tilde{u}^\varepsilon$ is bounded in $L^2$-norm and $|\nabla \psi^\varepsilon| \leq C \varepsilon^{-1}$, the first term on the right hand side tends to zero. Consider the second term. By (1.36) we have $\varepsilon \| \nabla \tilde{u}^\varepsilon \|_{L^2(\Omega_0^\varepsilon)} \leq C \| \tilde{u}^\varepsilon \|_{L^2(\Omega_0^\varepsilon)} \leq C$; then from the boundedness of $\nabla \psi_0 + b \nabla \varphi$ (in $L^\infty$-norm) we conclude that the second term also converges to zero. By Lemma 2.2.2 $\varepsilon \nabla u^\varepsilon$ weakly two-scale converges to $\nabla_y v(x,y)$, hence, by the definition of the weak two-scale convergence, we obtain

$$
\lim_{\varepsilon \to 0} \varepsilon^2 a_0 \int_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla \psi^\varepsilon \, dx = a_0 \int_{\Omega_1} \int_{Q_0} \nabla_y v(x,y) \varphi(x) \nabla_y b(y) \, dy \, dx.
$$

(2.20)

The third term on the left hand side of (2.18) converges to zero due to the smallness of the domain of integration. Indeed, since for small enough $\varepsilon$ the test function $\psi^\varepsilon$ is equal to $\psi_0$ on $\bar{\Omega}_0^\varepsilon$, $\| \tilde{a}_0^{1/2} \nabla u^\varepsilon \|_{L^2(\bar{\Omega}_0^\varepsilon)} \leq C$ uniformly in $\varepsilon$ (cf.
(1.25), and \(|\tilde{\Omega}_0| \to 0\) as \(\varepsilon \to 0\), we derive for small enough \(\varepsilon\)

\[
\left| \int_{\tilde{\Omega}_0} \tilde{a}_0 \nabla u^\varepsilon \nabla \psi^\varepsilon \, dx \right| \leq \int_{\tilde{\Omega}_0} \tilde{a}_0 \nabla u^\varepsilon \nabla \psi_0 \, dx \leq C \int_{\tilde{\Omega}_0} |\nabla u^\varepsilon| \, dx \leq C |\tilde{\Omega}_0|^{1/2} \tilde{a}_0^{1/2} \|\nabla u^\varepsilon\|_{L^2(\tilde{\Omega}_0)} \to 0. \tag{2.21}
\]

The eigenfunction \(u^\varepsilon\) coincides with \(\tilde{u}^\varepsilon\) on \(\Omega_2^\varepsilon\). Then, via (2.5) we have convergence of the last term on the left hand side of (2.18):

\[
\lim_{\varepsilon \to 0} a_2 \int_{\Omega_2} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx =
\]

\[
= \lim_{\varepsilon \to 0} \left[ a_2 \int_{\Omega_2} \nabla \tilde{u}^\varepsilon \cdot \nabla \psi_0 \, dx - a_2 \int_{\tilde{\Omega}_0 \cap \Omega_2} \nabla \tilde{u}^\varepsilon \cdot \nabla \psi_0 \, dx \right] = \int_{\Omega_2} \nabla u_0 \cdot \nabla \psi_0 \, dx, \tag{2.22}
\]

as

\[
\left| a_2 \int_{\tilde{\Omega}_0 \cap \Omega_2} \nabla \tilde{u}^\varepsilon \cdot \nabla \psi_0 \, dx \right| \leq C \int_{\tilde{\Omega}_0 \cap \Omega_2} |\nabla \tilde{u}^\varepsilon| \, dx \leq \|\nabla \tilde{u}^\varepsilon\| \tilde{\Omega}_0 \cap \Omega_2 \to 0.
\]

Now we will prove that the second term on the left hand side of (2.18) converges to the second term on the right hand side of (1.16) with \(u_0 = \psi_0\). We need to show that \(u^\varepsilon\) satisfies the conditions of Lemma 2.1.7. Let us show that convergence property (2.3) holds for \(u^\varepsilon\). To this end we substitute into (2.18) a test function of the form \(\psi^\varepsilon = \varepsilon w(\varepsilon^{-1}x)\varphi(x), \varphi \in C^\infty_0(\Omega_1), w \in C^\infty_{\text{per}}(\Box),\) cf. [48]. Since \(\nabla (\varepsilon w(\varepsilon^{-1}x)\varphi(x)) = O(1)\) and \(\|\varepsilon \nabla u^\varepsilon\|_{L^2(\Omega_0)}\) is bounded by (1.25) we have

\[
\left| \varepsilon^2 a_0 \int_{\Omega_0} \nabla u^\varepsilon \cdot \nabla (\varepsilon w\varphi) \, dx \right| \leq \varepsilon a_0 \|\nabla u^\varepsilon\|_{L^2(\Omega_0)} \|\nabla (\varepsilon w\varphi)\|_{L^2(\Omega_0)} \to 0.
\]
\[
\int_{\tilde{\Omega}_0} \tilde{a}_0 \nabla u^\varepsilon \cdot \nabla (\varepsilon w \varphi) \, dx = 0
\]
for small enough \( \varepsilon \) because \( \varphi \in C_0^\infty (\Omega_1) \) equals zero in \( \tilde{\Omega}_0 \) for small \( \varepsilon \) and, obviously,
\[
a_2 \int_{\tilde{\Omega}_2} \nabla u^\varepsilon \cdot \nabla (\varepsilon w \varphi) \, dx = 0.
\]
Since \( \varepsilon w \varphi = O(\varepsilon) \) as \( \varepsilon \to 0 \),
\[
\lambda_\varepsilon \int_{\mathbb{R}^n} u^\varepsilon w \varphi \, dx \to 0.
\]
Thus all the terms in (2.18) with \( \psi^\varepsilon = \varepsilon w \varphi \), except possibly
\[
a_1 \int_{\tilde{\Omega}_1} \nabla u^\varepsilon \cdot \nabla (\varepsilon w \varphi) \, dx = a_1 \int_{\tilde{\Omega}_1} \left[ \nabla u^\varepsilon \cdot \varepsilon w \nabla \varphi + \nabla u^\varepsilon (x) \cdot \nabla \psi (\varepsilon x) \varphi (x) \right] \, dx,
\]
converge to zero. Then the latter should also converge to zero. Since
\[
a_1 \int_{\tilde{\Omega}_1} \nabla u^\varepsilon \cdot \varepsilon w \nabla \varphi \, dx \to 0,
\]
we conclude the validity of (2.3).

The eigenfunction \( \tilde{u}^\varepsilon \) converges in the sense of the strong two-scale convergence and its gradient is bounded in \( L^2 \)-norm, see (2.8) and (1.28). Then by Lemma [2.1.6]
\[
\nabla \tilde{u}^\varepsilon \to \nabla u_0 (x) + \tilde{v} (x,y),
\]
where \( \tilde{v} \in L^2 (\mathbb{R}^n, V_{\text{pot}}) \). As long as \( \tilde{u}^\varepsilon \) coincides with \( u^\varepsilon \) on \( \Omega_1 \), we now can apply Lemma [2.1.7] to obtain
\[
\lim_{\varepsilon \to 0} a_1 \int_{\Omega_1} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx = \lim_{\varepsilon \to 0} a_1 \int_{\Omega_1} \nabla u^\varepsilon \cdot \nabla \psi_0 \, dx = \int_{\Omega_1} A^\text{hom} \nabla u_0 \cdot \nabla \psi_0 \, dx, \quad (2.23)
\]
where \( \psi^\varepsilon \) is as in (2.19).

Thus, passing to the limit as \( \varepsilon \to 0 \) on the left hand side of (2.18) via (2.20)—
(2.23), and on the right hand side via (2.17), we arrive at
\[
a_0 \int_{\Omega_1} \int_{Q_0} \nabla_y v \cdot \varphi \nabla_y b \, dy \, dx + \int_{\Omega_1} A^{\text{hom}} u_0 \cdot \nabla \psi_0 \, dx + a_2 \int_{\Omega_2} \nabla u_0 \cdot \nabla \psi_0 \, dx = \\
\lambda_0 \int_{\mathbb{R}^n} \int_{Q} (u_0 + v)(\psi_0 + \varphi b) \, dy \, dx.
\]
Since the space of functions from (2.19) is dense in \( V \) (see (1.17)), the latter is equivalent to (1.18). It follows from (2.17), Proposition 2.1.3 (ii) and the normalisation of \( u^\varepsilon \) that \( u_0(x, y) \neq 0 \). Thus we have proved that \( \lambda_0 \) and \( u_0(x, y) \) are respectively an eigenvalue and an eigenfunction of the operator \( A_0 \), completing the proof of the theorem.

**Remark 2.2.6.** Let \((a, b)\) be a gap in the spectrum of \( \hat{A}_0 \) and \( I \) be an interval lying strictly inside the gap. As we mentioned earlier, due to results of [48, 49] \( \sigma(\hat{A}_\varepsilon) \to \sigma(\hat{A}_0) \) in the sense of Hausdorff. This implies that for small enough \( \varepsilon \) the interval \( I \) belongs to the spectral gap of \( \hat{A}_\varepsilon \). Then we can implement Theorem 2 of [23] which claims that for large enough \( l \) small enough \( a_2 \), namely such that \( l^2/a_2 > C \), the operator \( A_\varepsilon \) with \( \Omega_2 = l\Omega \) has at least one localised eigenvalue \( \lambda_\varepsilon \) in \( I \). The constant \( C \) depends only on the size and position of \( I \) and geometric properties of \( \Omega \). Hence one can extract a converging subsequence \( \lambda_\varepsilon \) satisfying conditions of Theorem 2.2.1. Then from the latter follows the existence of eigenvalues of \( A_0 \) in the gaps of its essential spectrum, provided \( \Omega_2 \) is large enough and \( a_2 \) is small enough.

It is not hard to show that there holds the strong two-scale resolvent convergence \( A_\varepsilon \to A_0 \), see Definition 2.1.8. Consider the resolvent equation
\[
A_\varepsilon w^\varepsilon + \lambda w^\varepsilon = f^\varepsilon,
\]
where \(-\lambda \notin \sigma(A_0)\). It is well posed for small enough \( \varepsilon \) since for such \( \varepsilon \) \( \lambda \notin \sigma(A_\varepsilon) \). Suppose also that \( f^\varepsilon(x) \to f^0(x, y) \).

Multiplying this equation by \( w^\varepsilon \) and integrating by parts we obtain
\[
\|a_0^{1/2}(x, \varepsilon) \nabla w^\varepsilon\|_{L^2(\mathbb{R}^n)}^2 + \lambda \|w^\varepsilon\|_{L^2(\mathbb{R}^n)}^2 \leq \|w^\varepsilon\|_{L^2(\mathbb{R}^n)} \|f^\varepsilon\|_{L^2(\mathbb{R}^n)}.
\]
The weakly two-scale converging sequence \( f^\varepsilon \) is bounded in \( L^2 \). If \( \lambda \) is positive,
then we obviously get

\[ \|w^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C, \quad (2.25) \]

and

\[ \|a_0^{1/2}(x, \varepsilon)\nabla w^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C; \quad (2.26) \]

uniformly in \( \varepsilon \). If \( \lambda \) is negative, then \( \|w^\varepsilon\|_{L^2(\mathbb{R}^n)} \) could be bounded or unbounded. The case when \( \|w^\varepsilon\|_{L^2(\mathbb{R}^n)} \) is unbounded we will consider later. Otherwise we also have \((2.25), (2.26)\).

As when we considered the eigenvalue problem, we can represent the solution of \((2.24)\) as \( w^\varepsilon = \tilde{w}^\varepsilon + z^\varepsilon \), where \( \tilde{w}^\varepsilon \) is a harmonic extension of \( w^\varepsilon|_{\Omega_1\cup\Omega_2} \) to the whole \( \mathbb{R}^n \). Obviously \( \|\tilde{w}^\varepsilon\|_{L^2(\mathbb{R}^n)} \) and \( \|\nabla \tilde{w}^\varepsilon\|_{L^2(\mathbb{R}^n)} \) are bounded by \( \|w^\varepsilon\|_{L^2(\Omega_1\cup\Omega_2)} \) and \( \|\nabla w^\varepsilon\|_{L^2(\Omega_1\cup\Omega_2)} \). Then applying Proposition \((2.1.3) \quad (iv)\), Lemmas \(2.1.5\) and \(2.1.6\) we conclude that

\[ w^\varepsilon = \tilde{w}^\varepsilon + z^\varepsilon \xrightarrow{2} w^0(x, y) = w_0(x) + z(x, y) \in H^1(\mathbb{R}^n) + L^2(\Omega_1, H^1_{per}), \]

\[ \varepsilon \nabla z^\varepsilon(x) \xrightarrow{2} \nabla y z(x, y), \]

\[ \nabla \tilde{w}^\varepsilon(x) \xrightarrow{2} \nabla w_0(x) + v(x, y), \text{ where } v \in L^2(\mathbb{R}^n, V_{pot}). \]

As before, we can show that equality \((2.3)\) holds with \( u_\varepsilon = \tilde{w}^\varepsilon \), and then, applying Lemma \(2.1.7\) and the above convergence properties, pass to a limit in the weak form of \((2.24)\) with appropriately chosen test function to obtain

\[ A_0w^0 + \lambda w^0 = f^0. \]

Now suppose that \( f^\varepsilon \rightharpoonup f^0 \). We can carry out the same reasoning as in Lemma \(2.1.6\) (when we proved the strong two-scale convergence of \( v_\varepsilon^f \)) to prove that

\[ w^\varepsilon \overset{2}{\rightharpoonup} w^0. \]

In order to complete the proof of the strong two-scale resolvent convergence we need to consider the case when \( \lambda \) is negative and the sequence \( \|w^\varepsilon\|_{L^2(\mathbb{R}^n)} \) is unbounded. Then there is a subsequence \( w^\varepsilon \) with \( L^2 \)-norms converging to infinity. We divide equation \((2.24)\) by \( \|w^\varepsilon\|_{L^2(\mathbb{R}^n)} \) and rename \( \frac{w^\varepsilon}{\|w^\varepsilon\|_{L^2(\mathbb{R}^n)}} \) and \( \frac{f^\varepsilon}{\|w^\varepsilon\|_{L^2(\mathbb{R}^n)}} \) again as \( w^\varepsilon \) and \( f^\varepsilon \) to simplify the notation. Then we arrive at \((2.24)\) with \( \|w^\varepsilon\|_{L^2(\mathbb{R}^n)} = 1 \) and \( \|f^\varepsilon\|_{L^2(\mathbb{R}^n)} \to 0 \). By the properties of two-scale convergence \( f^\varepsilon \overset{2}{\rightharpoonup} 0 \). The by the above \( w^\varepsilon \overset{2}{\rightharpoonup} w^0 \), and \( \|w^0\|_{L^2(\mathbb{R}^n, Q)} = \|w^\varepsilon\|_{L^2(\mathbb{R}^n)} = 1 \), where \( w^0 \) satisfy the equation \( A_0w_0 + \lambda w_0 = 0 \). This means
that $-\lambda$ has to be an eigenvalue of $A_0$, which contradicts the initial assumption.

The strong two-scale resolvent convergence implies in particular the strong two-scale convergence of spectral projectors ($P_\varepsilon(\lambda) \xrightarrow{\varepsilon \to 0} P_0(\lambda)$ if $\lambda$ is not an eigenvalue of $A_0$), see [41, 48], and has other nice properties, however it does not imply in its own the convergence of the spectra. The latter requires an additional (two-scale) compactness property to hold, which Theorem 2.2.1 provides.

**Remark 2.2.7.** The function $v(x, y)$ could be represented as a product of $u_0(x)|_{\Omega_1}$ and $\lambda_0 b(y)$, where $b(y)$ solves (1.12) with $\lambda = \lambda_0$. Then $v(x, \varepsilon^{-1} x)$ strongly two-scale converges to $v(x, y)$ by the mean value property and the properties of two-scale convergence. Then

$$u_{\text{appr}}(x, \varepsilon) := \begin{cases} u_0(x) + v(x, x/\varepsilon), & x \in \Omega_0^*, \\ u_0(x), & x \in \mathbb{R}^n \setminus \Omega_0^*, \end{cases} \quad (2.27)$$

also strongly two-scale converges to $u^0(x, y)$. Hence it approximates the eigenfunction $u^\varepsilon(x)$:

$$\|u_{\text{appr}} - u^\varepsilon\|_{L_2(\mathbb{R}^n)}^2 = \int_{L_2(\mathbb{R}^n)} (u_{\text{appr}})^2 dx + \int_{L_2(\mathbb{R}^n)} (u^\varepsilon)^2 dx - 2 \int_{L_2(\mathbb{R}^n)} u_{\text{appr}} u^\varepsilon dx \to 0. \quad (2.28)$$

Using the result of Theorem 2.2.1 we will discuss the multiplicity properties of the eigenvalues $\lambda_\varepsilon$ and $\lambda_0$. Let us assume that the multiplicity of the eigenvalue $\lambda_0$ of $A_0$ is $m$. Suppose that for a subsequence $\varepsilon_k \to 0$ there exist $l$ (accounting for multiplicities) eigenvalues of $A_\varepsilon$, $\lambda_{\varepsilon_k,1} \leq \lambda_{\varepsilon_k,2} \leq \ldots \leq \lambda_{\varepsilon_k,l}$, such that $\lambda_{\varepsilon_k,i} \to \lambda_0, \ i = 1, \ldots, l$. Let $u^\varepsilon_{i,k}$ be the corresponding eigenfunctions orthonormalised in $L^2(\mathbb{R}^n)$. It follows from Theorem 2.2.1 that there exists a subsequence $k_m$ such that

$$u^\varepsilon_{i,k_m} \xrightarrow{\varepsilon \to 0} u^0_i, \ i = 1, \ldots, l,$$

where $u^0_i$ are eigenfunctions of $A_0$ corresponding to $\lambda_0$. In particular, due to the strong two-scale convergence, we have convergence of the inner products:

$$(u^\varepsilon_{i,k_m}, u^\varepsilon_{j,k_m})_{L^2(\mathbb{R}^n)} \to (u^0_i, u^0_j)_{\mathcal{H}_0}. \quad (2.29)$$

However $(u^\varepsilon_{i,k_m}, u^\varepsilon_{j,k_m})_{L^2(\mathbb{R}^n)} = \delta_{ij}$. Then $u^0_i, \ i = 1, \ldots, l$ are also orthonormal (in $\mathcal{H}_0$), i.e. there exist at least $l$ linearly independent eigenfunctions of $A_0$. 

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corresponding to $\lambda_0$. Thus, $l \leq m$.

The results presented in \[29\] remain also valid in our setting of the problem, i.e. when the coefficients of the divergence form operator $A_{\varepsilon}$ are of the form (1.4). By Theorem 4.1 of \[29\], if $\lambda_0$ is an eigenvalue of the limit operator $A_0$ lying in a gap of its essential spectrum, then for small enough $\varepsilon$, there exist eigenvalues (or at least one eigenvalue) of $A_{\varepsilon}$ such that

$$|\lambda_{\varepsilon,i} - \lambda_0| \leq C \varepsilon^{1/2}, i = 1, \ldots, l(\varepsilon). \quad (2.30)$$

Moreover, again by \[29\] Thm 4.1, for any eigenfunction $u_0^i$ of $A_0$ corresponding to $\lambda_0$ the related $u_{\varepsilon}^{i\text{ appr}}$, see (2.27), can be approximated by a linear combination of the eigenfunctions of $A_{\varepsilon}$ corresponding to $\lambda_{\varepsilon,i}$, $i = 1, \ldots, l(\varepsilon)$:

$$\|u_{\varepsilon}^{i\text{ appr}} - \sum_{j=1}^{l(\varepsilon)} c_{ij}(\varepsilon_k) u_{\varepsilon j}^i\|_{L^2(\mathbb{R}^n)} \leq C \varepsilon^{1/2}. \quad (2.31)$$

Then it is not hard to show that $l(\varepsilon) \geq m$. Assume, for contradiction, that it is not true. Then for some subsequence $\varepsilon_k$ we have

$$\|u_0^i - \sum_{j=1}^{l} c_{ij}(\varepsilon_k) u_{\varepsilon j}^i\|_{L^2(\mathbb{R}^n)} \leq C \varepsilon^{1/2}, \quad (2.32)$$

with $l < m$. Number of columns $l$ of the matrix $(c_{ij}(\varepsilon_k))$ is less than number of its rows $m$, so the latter are linearly dependent vectors, and there exist coefficients $\alpha_i(\varepsilon_k)$, $i = 1, \ldots, m$ not equal to zero simultaneously such that

$$\sum_{i=1}^{m} \alpha_i(\varepsilon_k) c_i(\varepsilon_k) = 0,$$

where $c_i(\varepsilon_k) = (c_{i1}(\varepsilon_k), \ldots, c_{il}(\varepsilon_k))$. Let coefficients $\alpha_i(\varepsilon_k)$ be normalised: $\sum_{i=1}^{m} |\alpha_i(\varepsilon_k)|^2 = 1$. It is obvious that then

$$\sum_{i=1}^{m} \alpha_i(\varepsilon_k) \sum_{j=1}^{l} c_{ij}(\varepsilon_k) u_{\varepsilon j}^i \equiv 0. \quad (2.32)$$
From (2.31) and (2.32) it follows that

\[
\left\| \sum_{i=1}^{m} \alpha_i(\varepsilon_k)u_i^{\text{appr}} \right\|_{L^2(\mathbb{R}^n)} = \left\| \sum_{i=1}^{m} \alpha_i(\varepsilon_k)(u_i^{\text{appr}} - \sum_{j=1}^{l} c_{ij}(\varepsilon_k)u_j^{\varepsilon_k}) \right\|_{L^2(\mathbb{R}^n)} \rightarrow 0.
\]

But on the other hand, by (2.29),

\[
\left\| \sum_{i=1}^{m} \alpha_i(\varepsilon_k)u_i^{\text{appr}} \right\|_{L^2(\mathbb{R}^n)}^2 = \sum_{i,k=1}^{m} \alpha_i(\varepsilon_k)\alpha_k(\varepsilon_k)(u_i^{\text{appr}}, u_k^{\text{appr}})_{L^2(\mathbb{R}^n)}
\]

\[= \sum_{i,k=1}^{m} \alpha_i(\varepsilon_k)\alpha_k(\varepsilon_k)(u_i^{\varepsilon_k}, u_k^{\varepsilon_k})_{H_0} + o(1) = \sum_{i,k=1}^{m} \alpha_i(\varepsilon_k)\alpha_k(\varepsilon_k)\delta_{ik} + o(1)
\]

\[= \sum_{i=1}^{m} |\alpha_i(\varepsilon_k)|^2 + o(1) \rightarrow 1.
\]

We get a contradiction. Thus, total multiplicity of \(\lambda(\varepsilon) \rightarrow \lambda_0\) is at least \(m\).

As a result we come to a conclusion that if \(\lambda_0\) is an eigenvalue of \(A_0\) of multiplicity \(m\) then there exist exactly \(m\) eigenvalues (counted with their multiplicities) of \(A_\varepsilon\) converging to \(\lambda_0\), and estimates (2.30) and (2.31) hold. In other words there is an “asymptotic one-to-one correspondence” between isolated eigenvalues and eigenfunctions of the operators \(A_\varepsilon\) and \(A_0\).

2.3 Identity of the essential spectra of \(\widehat{A}_0\) and \(A_0\), convergence of the spectra of \(A_\varepsilon\) in the sense of Hausdorff

We recall that \(\widehat{A}_\varepsilon\) and \(\widehat{A}_0\) denote the ‘unperturbed’ operators corresponding to \(A_\varepsilon\) and \(A_0\), see Section 1.1. It was shown in [18] that \(\sigma(\widehat{A}_\varepsilon) \xrightarrow{H} \sigma(\widehat{A}_0)\) (the spectra of both \(\widehat{A}_\varepsilon\) and \(\widehat{A}_0\) are purely essential). In [23] it is proved that the essential spectrum of a divergence form operator \(-\nabla \cdot a(x)\nabla\) (where \(a(x) \geq \delta > 0\) is a scalar function) remains unperturbed with respect to the local perturbation of the coefficient \(a(x)\). Applying this assertion to the operator \(\widehat{A}_\varepsilon\) and its perturbation \(A_\varepsilon\) we conclude that \(\sigma(\widehat{A}_\varepsilon) = \sigma_{\text{ess}}(A_\varepsilon) \xrightarrow{H} \sigma(\widehat{A}_0)\). Let us assume that \(\sigma(\widehat{A}_0) = \sigma_{\text{ess}}(A_0)\). Then \(\sigma_{\text{ess}}(A_\varepsilon) \xrightarrow{H} \sigma_{\text{ess}}(A_0)\). In this case Theorem 2.2.1 together with the results of [29] imply the convergence of the discrete spectra in the
gaps (σ_{disc}(A_ε) \xrightarrow{H} \sigma_{disc}(A_0)) and, consequently, we would have σ(A_ε) \xrightarrow{H} σ(A_0). However, we cannot apply the result of [23] as it is stated to the case of the two-scale operators \( \hat{A}_0 \) and \( A_0 \). In this section we prove the stability of the essential spectrum of \( \hat{A}_0 \) with respect to the local perturbation of its coefficients, establishing thereby the missing part of the reasoning. We do this by direct means using the Weyl’s criterion for the essential spectrum of an operator, see e.g. [12].

**Theorem 2.3.1.** The essential spectra of the operators \( \hat{A}_0 \) and \( A_0 \) coincide.

**Proof. Step 1.** First we describe the domains of \( \hat{A}_0 \) and \( A_0 \). According to the Friedrichs extension procedure, see e.g. [41], a function \( u \) belongs to \( D(A_0) \) if and only if \( u = u_0(x) + v(x,y) \in \mathcal{V} \) and there exists \( h = h_0(x) + g(x,y) \in \mathcal{H}_0 \) such that

\[
B_0(u, w) = (h, w)_{\mathcal{H}_0}
\]  

(2.33)

for all \( w = w_0 + z \in \mathcal{V} \), see (1.15)–(1.17).

Let \( u = u_0 + v \in D(A_0) \). Then in order to \( u_0 \in D(A_0) \) be fulfilled there must be a function \( f \in \mathcal{H}_0 \) such that

\[
a_2 \int_{\Omega_2} \nabla u_0 \cdot \nabla w_0 \, dx + \int_{\Omega_1} A_{\text{hom}} \nabla u_0 \cdot \nabla w_0 \, dx = (f, w_0 + z)_{\mathcal{H}_0}
\]  

(2.34)

for all \( w \in \mathcal{V} \). In particular, setting in (2.33) \( z \equiv 0 \) we obtain

\[
a_2 \int_{\Omega_2} \nabla u_0 \cdot \nabla w_0 \, dx + \int_{\Omega_1} A_{\text{hom}} \nabla u_0 \cdot \nabla w_0 \, dx = \int_{\mathbb{R}^n} w_0(h_0 + \langle g \rangle) \, dy \, dx.
\]  

(2.35)

Comparing (2.34) and (2.35) we infer that their right hand sides are equal and that \( f \) is orthogonal to \( L^2(\Omega_1; L^2(Q_0)) \). One can derive that \( f \) satisfying (2.34) is defined by

\[
f = \begin{cases} 
  h_0, & x \in \Omega_2, \\
  |Q_1|^{-1} (h_0 + \langle g \rangle) \Theta_{Q_1}(y), & x \in \Omega_1.
\end{cases}
\]

Therefore \( u_0 \) and, hence, \( v \) belong to \( D(A_0) \) as soon as \( u = u_0 + v \in D(A_0) \). Due to the regularity properties of solutions of elliptic equations, \( u_0 \in H^2_{\text{loc}} \) everywhere away from the boundary of \( \Omega_2 \).

Operator \( \hat{A}_0 \) acting in the Hilbert space \( \hat{\mathcal{H}}_0 \) was described in [48] and is generated by a (closed) symmetric and bounded from below bilinear form \( \hat{B}_0(u, w) \) on a dense subspace \( \hat{\mathcal{V}} \) of \( \hat{\mathcal{H}}_0 \), where \( \hat{\mathcal{H}}_0 \), \( \hat{\mathcal{V}} \) and \( \hat{B}_0(u, w) \) are defined by (1.7)–(1.9). A function \( u \) belongs to domain \( D(\hat{A}_0) \) if and only if \( u = u_0(x) + v(x,y) \in \hat{\mathcal{V}} \).
and there exists $h \in \hat{\mathcal{H}}_0$ such that

$$\hat{B}_0(u, w) = (h, w)_{\hat{\mathcal{H}}_0}$$

for all $w \in \hat{V}$. Analogously to the case of $\mathcal{D}(A_0)$, if $u = u_0 + v \in \mathcal{D}(\hat{A}_0)$ then $u_0, v \in \mathcal{D}(\hat{A}_0)$, $u_0 \in H^2(\mathbb{R}^n)$.

Let $A$ be a self-adjoint operator with domain $\mathcal{D}(A)$ acting in a Hilbert space $H$. By the Weyl’s criterium, see e.g. [12], condition $\lambda \in \sigma_{ess}(A)$ is equivalent to the existence of a singular sequence $u^{(k)} \in \mathcal{D}(A)$, i.e. such that

$$0 < C_1 \leq \|u^{(k)}\|_H \leq C_2,$$  \hspace{1cm} (2.36)

$$u^{(k)} \rightharpoonup 0 \text{ weakly in } H,$$  \hspace{1cm} (2.37)

$$(A - \lambda)u^{(k)} \to 0 \text{ strongly in } H.$$  \hspace{1cm} (2.38)

Employing this definition we will prove that $\lambda \in \sigma_{ess}(A_0)$ if and only if $\lambda \in \sigma_{ess}(\hat{A}_0)$. The operators $A_0$ and $\hat{A}_0$ possess very similar properties. The main difference between them consists in the fact that their domains differ. Luckily, a function which support does not intersect with $\Omega_2$ belongs to $\mathcal{D}(A_0)$ and $\mathcal{D}(\hat{A}_0)$ simultaneously. So the idea of the proof is the following. We consider arbitrary singular sequence of one operator and change it slightly to ensure that its elements belong the domain of another operator preserving all properties (2.36)–(2.38).

**Step 2.** Let $\lambda \in \sigma_{ess}(\hat{A}_0)$ and $u^{(k)} = u_0^{(k)}(x) + v^{(k)}(x, y)$ be the corresponding singular sequence in $\mathcal{D}(\hat{A}_0) \subset \hat{\mathcal{H}}_0$. First notice that the gradient of $u_0^{(k)}$ is bounded in $L^2(\mathbb{R}^n)$. Indeed, from (1.9) and (2.38) we have

$$\|\nabla u_0^{(k)}\|_{L^2(\mathbb{R}^n)} \leq C\hat{B}_0(u^{(k)}, u^{(k)}) = C\lambda(u^{(k)}, u^{(k)})_{\hat{\mathcal{H}}_0} + o(1) \leq C.$$  \hspace{1cm} (2.39)

Let us define a cut-off function

$$\eta_{k,R}(x) = \eta \left( \frac{1}{k}(|x| - R) \right),$$

where $\eta \in C^2(\mathbb{R})$ is such that

$$\eta(t) = \begin{cases} 
1, & t \leq 0, \\
0, & t \geq 1.
\end{cases}$$

So $\eta_{k,R}$ is 1 when $|x| \leq R$, 0 when $|x| \geq R + k$ and has small gradient if $k$ is
large.

Consider the following sequence, \( u^{(k)} \eta_{k,R_k} \in D(\hat{A}_0) \), where \( R_k \) is chosen large enough so that \( \|u^{(k)}(1 - \eta_{k,R_k})\|_{\tilde{H}_0} \leq \frac{1}{k} \). This sequence obviously satisfies (2.36) for large enough \( k \) regarding the operator \( \hat{A}_0 \). Let us check property (2.38). The operator \( \hat{A}_0 \) acts on a function \( u \in H^2(\mathbb{R}^n) \subset D(\hat{A}_0) \) as follows\(^4\), cf. [48]. Let
\[
-\nabla \cdot A^{\text{hom}} \nabla u(x) = f(x) \in L^2(\mathbb{R}^n).
\]
Then, by the definition of \( \hat{A}_0 \), we have
\[
\hat{A}_0 u(x) = |Q_1|^{-1} \Theta Q_1(y) f(x) \in \hat{H}_0.
\]
Note that
\[
\|\hat{A}_0 u\|_{\tilde{H}_0} = |Q_1|^{-1/2} \|f\|_{L^2(\mathbb{R}^n)}.
\]
For \( u^{(k)} \eta_{k,R_k} \) we derive
\[
\hat{A}_0 \left( u^{(k)} \eta_{k,R_k} \right) = \eta_{k,R_k} \hat{A}_0 u^{(k)} - |Q_1|^{-1} \Theta Q_1(y) \left( 2\nabla \eta_{k,R_k} \cdot A^{\text{hom}} \nabla u_0^{(k)} + u_0^{(k)} \nabla \cdot A^{\text{hom}} \nabla \eta_{k,R_k} \right).
\]
The second term on the right hand side becomes small as \( k \to \infty \). Thus we arrive at
\[
\left\| (\hat{A}_0 - \lambda)(u^{(k)} \eta_{k,R_k}) \right\|_{\tilde{H}_0} \leq \left\| \eta_{k,R_k} (\hat{A}_0 - \lambda) u^{(k)} \right\|_{\tilde{H}_0} + 2|Q_1|^{-1/2} \left\| \nabla \eta_{k,R_k} \cdot A^{\text{hom}} \nabla u_0^{(k)} \right\|_{L^2(\mathbb{R}^n)} + |Q_1|^{-1/2} \left\| u_0^{(k)} \nabla \cdot A^{\text{hom}} \nabla \eta_{k,R_k} \right\|_{L^2(\mathbb{R}^n)} = O(1) + \frac{1}{k^2} O \left( \|\nabla u_0^{(k)}\|_{L^2(\mathbb{R}^n)} \right) + \frac{1}{k^2} O \left( \|u_0^{(k)}\|_{L^2(\mathbb{R}^n)} \right).
\]
Due to (2.36) and (2.39) the latter converges to 0 as \( k \to \infty \). Hence (2.38) holds regarding \( \hat{A}_0 \) and \( u^{(k)} \eta_{k,R_k} \).

Now notice that if \( \text{supp} \, u \cap \Omega_2 = \emptyset \), then \( u \in D(\hat{A}_0) \) if and only if \( u \in D(A_0) \); besides \( \hat{A}_0 u = A_0 u \). We hence next shift the supports of \( u^{(k)} \eta_{k,R_k} \) away from \( \Omega_2 \) ensuring also that the new sequence is weakly convergent to maintain (2.37).

---

\(^{4}\) If \( u = u_0(x) + v(x,y) \) then \( \hat{A}_0 u = h \in \tilde{H}_0 \) implies \( -\nabla \cdot A^{\text{hom}} \nabla u_0 = \langle h \rangle_y \) and \( -a_0 \Delta_y v = h(x,y), y \in Q_0 \).
Since $\text{supp}\eta_{k,R_k}$ is a closed ball of radius $R_k + k$ centred at the origin, the shift of $x$ by $\xi_k := (R_k + 2k + \text{diam}(\Omega_2))\xi$ for every $k$, where $\xi$ is an arbitrary unit vector from $\mathbb{R}^n$, will do the job. Hence, for the given $\lambda$ we have constructed a singular sequence

$$w^{(k)}(x, y) = u^{(k)}(x + \xi_k, y) \eta_{k,R_k}(x + \xi_k),$$

satisfying all the properties (2.36)–(2.38) for the operator $A_0$. Namely, the translational invariance of $\hat{A}_0$ in $x$ ensures that (2.36) and (2.38) are satisfied. Finally, (2.37) follows from the pointwise convergence of $w^{(k)}$ to zero as $k \to \infty$ (since for any fixed $x$, $w^{(k)}(x, y) = 0$ for large enough $k$). Thus $\lambda \in \sigma_{\text{ess}}(A_0)$.

**Step 3.** Suppose now that $\lambda \in \sigma_{\text{ess}}(A_0)$ and $u^{(k)} = u_0^{(k)}(x) + v^{(k)}(x, y)$ is the corresponding singular sequence. Let $R$ be such that $\overline{\Omega}_2 \subset B_R$. The situation now is more complicated. The elements of the sequence does not belong to $\mathcal{D}(\hat{A}_0)$ because of the discontinuity of first derivative at the boundary of $\Omega_2$. If we cut off the elements of the sequence in the neighbourhood of $\Omega_2$ we may lose property (2.36). This may happen when functions $u^{(k)}$ mainly “concentrated” around $\Omega_2$. In fact there are two possibilities: either functions $u^{(k)}$ decay uniformly at infinity in $\mathcal{H}_0$ or not. In the first case it is possible to prove the compactness of $u_0^{(k)}$ in $L^2(\mathbb{R}^n)$. Due to (2.37) the latter implies $u_0^{(k)} \to 0$. Then the sequence $v^{(k)}$ satisfies all the properties of Weyl sequence for $\hat{A}_0$ and belongs its domain. In the second case we can cut off $u^{(k)}$ in the neighbourhood of $\Omega_2$ to obtain Weyl sequence straight away. In the following we carry out this sketch in more precise way.

There are only two alternative possibilities:

- There exists a sequence $\delta_i \to 0$ such that for any $i \in \mathbb{N}$

  $$\|u^{(k)}(1 - \Theta_{B_{R+k+i}})\|_{\mathcal{H}_0} \leq \delta_i \quad (2.41)$$

  for all $k$.

- There exist a constant $M > 0$ and subsequences $k(j) \to \infty$, $i(j) \to \infty$ as $j \to \infty$ such that

  $$\|u^{(k(j))}(1 - \Theta_{B_{R+k(j)+i(j)}})\|_{\mathcal{H}_0} \geq M \quad (2.42)$$

\footnote{Let $A_{k_i} := \|u^{(k)}(1 - \Theta_{B_{R+k_i}})\|_{\mathcal{H}_0}$ and let $\delta_i := \sup_k A_{k_i}$. Then either $\delta_i \to 0$ giving (2.41) or $\delta_i \not\to 0$ yielding (2.42).}
for all \( j \).

Let (2.41) take place. The sequence \( \nabla u_0^{(k)} \) is bounded in \( L^2(\mathbb{R}^n) \), cf. (2.39). From (2.41) and

\[
\|f\|_{L^2(\mathbb{R}^n)} = \|f\|_{\mathcal{H}_0}, \quad \text{for all} \ f \in L^2(\mathbb{R}^n) \subset \mathcal{H}_0, \tag{2.43}
\]

it follows that

\[
u_0^{(k)} \to u(x) \text{ in } L^2(\mathbb{R}^n), \tag{2.44}
\]

up to a subsequence. The reasoning leading to this assertion is essentially identical to the one in (2.4)–(2.7) and is not reproduced here. Since \( u^{(k)} = u_0^{(k)} + v^{(k)} \) converges weakly in \( \mathcal{H}_0 \) to zero, from (2.44) we conclude that

\[
\forall (x,y) \in \mathbb{R}^n \setminus Q_0, \quad v^{(k)}(x,y) \rightharpoonup -u(x) \text{ weakly in } \mathcal{H}_0.
\]

Hence, on one hand, we have

\[
(u, v^{(k)})_{\mathcal{H}_0} \to -(u,u)_{\mathcal{H}_0} = -\int_{\mathbb{R}^n} u^2 \, dx
\]
as \( k \to \infty \). On the other hand,

\[
(u, v^{(k)})_{\mathcal{H}_0} = \int_{\mathbb{R}^n} \int_{Q_0} u v^{(k)} \, dy \, dx = \int_{\mathbb{R}^n} \int_{Q_0} u \Theta_{Q_0}(y) v^{(k)} \, dy \, dx = (u \Theta_{Q_0}(y), v^{(k)})_{\mathcal{H}_0} \to - (u \Theta_{Q_0}(y), u)_{\mathcal{H}_0} = -|Q_0| \int_{\mathbb{R}^n} u^2 \, dx.
\]

Comparing the last two formulas, we conclude that \( u \equiv 0 \), i.e.

\[
u_0^{(k)} \to 0 \text{ in } L^2(\mathbb{R}^n). \tag{2.45}
\]

Moreover

\[
v^{(k)}(x,y) \to 0 \text{ weakly in } \mathcal{H}_0. \tag{2.46}
\]

Let us consider an arbitrary sequence \( g^{(k)} = g_0^{(k)} + h^{(k)} \) from \( \mathcal{H}_0 \) converging to zero. It is simple to prove, but probably not entirely obvious that both \( g_0^{(k)} \)
and \( h^{(k)} \) converge to zero. We can write the terms as \( g^{(k)} = g_0^{(k)} \Theta Q_1(y) + (g_0^{(k)} \Theta Q_0(y) + h^{(k)}) \). We obtain

\[
\|g^{(k)}\|_{\mathcal{H}_0}^2 = |Q_1| \int_{\mathbb{R}^n} (g_0^{(k)})^2 \, dx + \int_{\mathbb{R}^n} \int_{Q_0} (g_0^{(k)} \Theta Q_0(y) + h^{(k)})^2 \, dy \, dx \to 0.
\]

Then it follows that \( g_0^{(k)} \) converges to zero (in \( L^2(\mathbb{R}^n) \) and \( \mathcal{H}_0 \)), and, hence, \( h^{(k)} \) converges to zero (in \( \mathcal{H}_0 \)).

Now we denote \( A_0 u^{(k)} \) by \( g^{(k)}(x,y) = g_0^{(k)}(x) + h^{(k)}(x,y) \in \mathcal{H}_0 \). From (2.38) we get the following convergence:

\[
\|g_0^{(k)} - \lambda u_0^{(k)}\|_{L^2(\mathbb{R}^n)} \to 0,
\]
\[
\|h^{(k)} - \lambda v^{(k)}\|_{\mathcal{H}_0} \to 0. \tag{2.47}
\]

Then (2.45) implies that

\[
g_0^{(k)} \to 0 \text{ in } L^2(\mathbb{R}^n). \tag{2.48}
\]

One might expect now that \( v^{(k)} \) has to be a Weyl sequence for the operator \( A_0 \) (and also for the operator \( \widehat{A}_0 \), as \( v^{(k)} \) extended by zero into \( \Omega_2 \) belongs to its domain). However it is not true. Functions \( v^{(k)} \) satisfy the following equation

\[
A_0 v^{(k)} = g_0^{(k)} \Theta \Omega_1(x) \Theta Q_0(y) + h^{(k)} - |Q_1|^{-1} \Theta Q_1(y) \left( g_0^{(k)} \Theta \Omega_1(x) \Theta Q_0(y) + h^{(k)} \right)_y.
\]

Substituting the expression on the right hand side into \( \|A_0 v^{(k)} - \lambda v^{(k)}\|_{\mathcal{H}_0} \) one finds that this entity does not converge to zero. Nevertheless, \( v^{(k)} \) turns out to be a Weyl sequence for an operator \( \widehat{A}_y \), see below, whose spectrum is contained in the essential spectrum of \( \widehat{A}_0 \),

\[
\sigma(\widehat{A}_y) \subset \sigma_{\text{ess}}(\widehat{A}_0), \tag{2.49}
\]

see [49]. We define a self-adjoint operator \( \widehat{A}_y \) (cf. [48]) acting in \( L^2(\Omega_1 \times Q_0) \) by

\[
\widehat{A}_y v = -a_0 \Delta_y v = p, \quad p \in L^2(\mathbb{R}^n \times Q_0).
\]

The domain of the operator, \( \mathcal{D}(\widehat{A}_y) \subset L^2(\mathbb{R}^n, H_0^1(Q_0)) \), is the set of all the solution of this equation. It is not difficult to see (by analysing (1.16)) that

\[
\widehat{A}_y v^{(k)} = g_0^{(k)} \Theta \Omega_1(x) \Theta Q_0(y) + h^{(k)}, \tag{2.50}
\]
i.e. \( v^{(k)} \in \mathcal{D}(\hat{A}_y) \). Combining (2.47), (2.48) and (2.50) we arrive at

\[
\| (\hat{A}_y - \lambda) v^{(k)} \|_{L^2(\mathbb{R}^n \times Q_0)} = \| h^{(k)}_0 \Theta Q_0 (y) + h^{(k)} - \lambda v^{(k)} \|_{L^2(\mathbb{R}^n \times Q_0)} \to 0.
\]

From (2.45) and (2.46) we conclude that other properties of Weyl sequence are fulfilled, and hence \( \lambda \in \hat{A}_y \). Hence \( \lambda \in \sigma_{\text{ess}}(\hat{A}_0) \), see (2.49).

Now let (2.42) hold. Consider a sequence \( w^{(j)} = u^{(k(j))}(1 - \eta_{(j), R}) \in \mathcal{D}(\hat{A}_0) \) (we remind that \( R \) is large enough to ensure \( \Omega \subset B_R \)). Then

\[
\| w^{(j)} \|_{\hat{H}_0} \geq \| u^{(k(j))}(1 - \Theta_{B_{R+1}(j)}) \|_{\mathcal{H}_0} \geq M,
\]

i.e. (2.36) is satisfied for \( w^{(j)} \). Since the sequence \( 1 - \eta_{(j), R} \) tends to 0 pointwise, (2.37) is valid. Analogously to (2.40) we derive

\[
\| (\hat{A}_0 - \lambda) w^{(j)} \|_{\hat{H}_0} = \| (A_0 - \lambda) w^{(j)} \|_{\mathcal{H}_0} \to 0,
\]

yielding (2.38). Thus, we conclude that \( \lambda \in \sigma_{\text{ess}}(\hat{A}_0) \), completing the proof of the theorem. \( \square \)

**Remark 2.3.2.** Theorem 2.3.1 combined with [48] implies that \( \sigma_{\text{ess}}(A_0) = \{ \lambda : \beta(\lambda) \geq 0 \} \cup \sigma(A_y) \). Using the methods of [48] it is not hard to show further that \( \sigma_{\text{ess}}(A_0) \) contains no point spectrum (in particular, no embedded eigenvalues) except if \( \lambda \) is an eigenvalue of \( A_y \) corresponding to an eigenfunction with zero mean. It is natural to conjecture (cf. [48]) that, outside these eigenvalues, the spectrum is absolutely continuous and the “eigenfunctions of the continuous spectrum” are \( u(x, y, \lambda) = u_0(x, \lambda)(1 + \lambda b(y, \lambda)) \), where \( u_0(x, \lambda) \) are solutions of the appropriate scattering problems:

\[
\nabla \cdot A_{\text{hom}} \nabla u_0 + \beta(\lambda) u_0 = 0, \quad x \in \mathbb{R}^n \setminus \Omega_2,
\]

\[
a_2 \Delta u_0 + \lambda u_0 = 0, \quad x \in \Omega_2 \tag{2.51}
\]

with the appropriate matching condition at \( \partial \Omega_2 \) and radiation condition at infinity. A detailed study of this as well as of the convergence of the related generalised eigenfunctions (cf. [48] for the defect-free case) is beyond the scope of the present study.

Summarising the main results of the chapter we conclude that Theorems 2.2.1 and 2.3.1 together with the results of 23, 29 (see the discussions at the end of
Section 2.2 and in the beginning of the present section) establish the validity of Theorem 1.1.1.
Part II

Spectral asymptotics in networks of thin domains
Chapter 3

Asymptotics of eigenfunctions and eigenvalues

This chapter is devoted to the construction of the asymptotics of the eigenvalue problem for the Laplacian in a thin curved domain $\Omega_h$ with Neumann boundary condition on one slanted end and Dirichlet condition elsewhere. We first state the problem and make a change of variables so as to pass to a problem in a fixed rectangle for a differential operator formally given by an asymptotic series. We then formally construct the asymptotics of the eigenvalues and eigenfunctions in Section 3.2 (outer problem), where the main order terms are functions of separated variables - transversal and longitudinal. In order to obtain proper boundary conditions for the functions of longitudinal variable we use the method of matched asymptotic expansions. Namely, we match the asymptotics of the outer problem with the asymptotics of the solution to the inner problem, which reveals the behaviour of the eigenfunctions of the problem in $\Omega_h$ in a small neighbourhood of the slanted end. The solution to the inner problem is described by means of scattering theory. In the last section of this chapter we provide the justification of the derived asymptotics and obtain relevant error bounds.

3.1 Problem formulation

We consider an eigenvalue problem for the Laplacian in a thin curved strip $\Omega_h$ with a slanted edge described as follows. Let $\Gamma$ be a smooth curve in $\mathbb{R}^2$ with a natural parametrisation $r(s) = (r_1(s), r_2(s))^T$, $s \in [0, 1]$. The length of tangential vector $r'(s)$ is one. The unit normal vector is given by $n(s) = (-r_2'(s), r_1'(s))^T$. We assume that $r(0) = 0$ and the curvature is zero in
some neighbourhood of zero, say \( s \in [0, s_0], s_0 < 1 \), for definiteness, and that \( r_1(s) = s, \ r_2(s) = 0, \ s \in [0, s_0] \), i.e. \( \Gamma \) coincides with the positive part of \( x_1 \) axis in this neighbourhood. Let \( n \) be a normal coordinate along \( n(s) \). Then, for small enough \( h > 0 \) we define a thin curved strip \( \Omega_h \) by

\[
\Omega_h := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r_1(s) - nr_2'(s) \\ r_2(s) + nr_1'(s) \end{pmatrix} \right| s \in (0, 1), n \in (0, h), n < \tan(\alpha) s \right\},
\]

(3.1)

where \( 0 < h \ll 1 \) is a small parameter and \( 0 < \alpha \leq \pi/2 \) is some fixed angle describing the slant of the left edge, see Figure 3-1. We denote by \( \gamma_1 \) the part of the boundary of \( \Omega_h \) that is described by the equation \( n = \tan(\alpha)s \). Respectively \( \gamma_2 = \partial\Omega_h \setminus \gamma_1 \).

We study the following spectral problem:

\[
-\Delta u_h = \lambda_h u_h, \quad x \in \Omega_h,
\]
\[
\frac{\partial u}{\partial \nu} = 0, \quad x \in \gamma_1,
\]
\[
u u = 0, \quad x \in \gamma_2,
\]

(3.2)

where \( \nu \) is an exterior unit normal to the boundary of \( \Omega_h \). Denote the corresponding self-adjoint operator by \( A_h \). We are interested in finding an asymptotic solution to the problem. The small parameter \( h \) describes the thickness of the domain \( \Omega_h \), i.e. the shape of the domain changes with \( h \). We next aim at changing the variables so that the transformed spectral problem is in a fixed domain.
Let us rewrite the Laplacian in coordinates \((s, n)\). First we write the partial derivatives with respect to \(s\) and \(n\):

\[
\frac{\partial}{\partial s} = (r'_1 - nr''_2) \frac{\partial}{\partial x_1} + (r'_2 + nr''_1) \frac{\partial}{\partial x_2},
\]

\[
\frac{\partial}{\partial n} = -r'_2 \frac{\partial}{\partial x_1} + r'_1 \frac{\partial}{\partial x_2}.
\]  \(\text{(3.3)}\)

Since the curve’s parametrisation is natural, the vector \(\mathbf{r}''\) is normal to the curve (and hence parallel to \(\mathbf{n}\)). In this case the curvature is usually defined as the length of \(\mathbf{r}''\). However in order to operate with the notation more conveniently we define the curvature with sign:

\[
\kappa = \mathbf{n} \cdot \mathbf{r}''.
\]  \(\text{(3.4)}\)

We assume that \(\kappa \in C^2[0, 1]\). Obviously, \(\kappa \mathbf{n} = \mathbf{r}''\). This implies that \(r''_1 = -\kappa r'_2\) and \(r''_2 = \kappa r'_1\). Substituting the latter into \((3.3)\) we arrive at

\[
\begin{pmatrix}
\frac{\partial}{\partial s} \\
\frac{\partial}{\partial n}
\end{pmatrix} = \begin{pmatrix} A r'_1 & A r'_2 \\
-r'_2 & r'_1
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2}
\end{pmatrix},
\]

where \(A = 1 - \kappa n\). We inverse matrix on the right to obtain

\[
\begin{pmatrix}
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2}
\end{pmatrix} = \frac{1}{A} \begin{pmatrix} r'_1 & -A r'_2 \\
r'_2 & A r'_1
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial s} \\
\frac{\partial}{\partial n}
\end{pmatrix}.
\]

Therefore

\[
\frac{\partial^2}{\partial x_1^2} = A^{-1} r'_1 \left( \frac{r''_1}{A} + \frac{\kappa' r'_1}{A^2} \right) \frac{\partial}{\partial s} + \frac{(r'_1)^2}{A^2} \frac{\partial^2}{\partial s^2} - \frac{r'_1 r''_2}{A} \frac{\partial}{\partial n} - \frac{\kappa r'_1 r''_2}{A^2} \frac{\partial}{\partial s} - 2 \frac{r'_1 r''_2}{A} \frac{\partial}{\partial s} \frac{\partial}{\partial n} + (r'_2)^2 \frac{\partial^2}{\partial n^2},
\]
and
\[
\frac{\partial^2}{\partial x_2^2} = A^{-1} r_2' \left( \frac{r_2''}{A} + \frac{\kappa' \kappa r_2'}{A^2} \right) \frac{\partial}{\partial s} + \frac{(r_2')^2}{A^2} \frac{\partial^2}{\partial s^2} + \frac{r_2' r_2''}{A} \frac{\partial}{\partial n} + \frac{\kappa r_2' r_2''}{A} \frac{\partial}{\partial s} \frac{\partial}{\partial n} + \frac{(r_2')^2}{A^2} \frac{\partial^2}{\partial n^2}.
\]

From the relations \( r' \cdot r'' = 0 \), \( (r')^2 = 1 \) and (3.4) we derive
\[
\Delta = \kappa' n \frac{\partial}{\partial s} + \frac{1}{A^2} \frac{\partial^2}{\partial s^2} - \frac{\kappa}{A} \frac{\partial}{\partial n} + \frac{\partial^2}{\partial n^2}.
\]

We can rewrite this in a more convenient form
\[
\Delta = (1 - \kappa n)^{-1} \frac{\partial}{\partial s} (1 - \kappa n)^{-1} \frac{\partial}{\partial s} + (1 - \kappa n)^{-1} \frac{\partial}{\partial n} (1 - \kappa n) \frac{\partial}{\partial n}. \quad (3.5)
\]

We will seek an asymptotic solution of (3.2). To obtain an asymptotic approximation of the eigenvalue problem with respect to the small parameter \( h \) we introduce the following rescaling:
\[
\eta = \frac{n}{h},
\]
and consider an eigenvalue problem in the rectangular domain of variables \((s, \eta)\),
\[
D = (0, 1) \times (0, 1).
\]

We use the Taylor’s expansion \((1 - h \kappa \eta)^{-1} = 1 + h \kappa \eta + (h \kappa \eta)^2 + \ldots \) and \( \partial / \partial n = h^{-1} \partial / \partial \eta \) to write a formal asymptotic expansion of the Laplacian:
\[
\begin{align*}
-\Delta &= -h^{-2} \frac{\partial^2}{\partial \eta^2} + h^{-1} \kappa \frac{\partial}{\partial \eta} + \left( \kappa^2 \eta \frac{\partial}{\partial \eta} - \frac{\partial^2}{\partial s^2} \right) + \\
&\quad + h \left( \kappa^2 \eta \frac{\partial}{\partial \eta} - 2 \kappa \eta \frac{\partial^2}{\partial s^2} - \kappa' \eta \frac{\partial}{\partial s} \right) + O(h^2) = -\Delta_h + O(h^2).
\end{align*}
\]

### 3.2 Outer problem: asymptotic expansions

In this section we seek a formal asymptotic solution to the eigenvalue problem for the operator \(-\Delta_h\) in a rectangular domain \(D\):
\[
- \Delta_h u_h = \lambda_h u_h, \quad (s, \eta) \in D, \quad (3.7)
\]
satisfying Dirichlet boundary condition \( u_h = 0 \) on the part of \( \partial D \) corresponding to \( \eta = 0, \eta = 1 \) and \( s = 1 \). We do not specify any particular boundary condition at \( s = 0 \) at the moment. The problem of finding a correct boundary condition at this part of the boundary is one of the main goals of the present chapter and requires a considerable special attention.

Due to the structure of \( -\Delta_h \) it is natural to seek the asymptotic solution to the spectral problem in the form of a standard regular asymptotic expansion:

\[
  u_h \approx u_0(s, \eta) + hu_1(s, \eta) + h^2u_2(s, \eta) + h^3u_3(s, \eta) + \ldots \tag{3.8}
\]

\[
  \lambda = \lambda_h \approx h^{-2}\lambda_{-2} + h^{-1}\lambda_{-1} + \lambda_0 + h\lambda_1 + \ldots \tag{3.9}
\]

We substitute (3.8), (3.9) into (3.7) and collect terms at the equal powers of \( h \), obtaining a recurrent sequence of differential equations, as follows.

\( h^{-2} \):

\[
  -\frac{\partial^2}{\partial \eta^2} u_0 = \lambda_{-2} u_0. \tag{3.10}
\]

The variable \( s \) in this equation plays the role of a parameter. This, together with the boundary conditions, implies

\[
  u_0 = \varphi_0(\eta)v_0(s),
\]

\[
  \varphi_0 = \sin(\pi \eta),
\]

\[
  \lambda_{-2} = \pi^2. \tag{3.11}
\]

Here \( v_0 \) is some function which will be defined at later stages. Notice that we restrict our attention to the eigenvalues \( \lambda_h \) corresponding to the first transversal mode \( \pi^2 \), the eigenvalues \( \lambda_h \) ’produced’ by the transversal modes \( n^2\pi^2, n = 2, 3, \ldots \) are beyond the scope of the present work.

\( h^{-1} \):

\[
  -\frac{\partial^2u_1}{\partial \eta^2} - \lambda_{-2} u_1 = -\kappa \frac{\partial}{\partial \eta} u_0 + \lambda_{-1} u_0 = (-\varphi_0'\kappa + \lambda_{-1} \varphi_0) v_0. \tag{3.12}
\]

As above, \( s \) is a parameter. This problem is solvable if and only if the right hand
side is orthogonal to the eigenfunction $\varphi_0$. So we obtain

$$
\int_0^1 (-\varphi_0'\kappa + \lambda\varphi_0) \varphi_0 \, d\eta = -\frac{\kappa v_0}{2} \int_0^1 \frac{d}{d\eta} \sin^2(\pi\eta) \, d\eta + \\
+\lambda v_0 \int_0^1 \sin^2(\pi\eta) \, d\eta = \lambda v_0 \int_0^1 \sin^2(\pi\eta) \, d\eta = 0,
$$

from which, assuming $v_0 \neq 0$, it follows that

$$
\lambda = 0.
$$

A general solution to (3.12) can be presented as a sum of the general solution of the homogeneous equation and some solution of the inhomogeneous equation. So,

$$
u_1 = \varphi_1(\eta)v_1(s) + \varphi_0(\eta)w_0(s),
$$

$$
v_1 = \kappa v_0,
$$

$$
\varphi_1 = \frac{1}{2} \eta \varphi_0,
$$

where $w_0$ is some function which will be defined at a later stage. We will obtain an equation for $v_0$ at the next step from the solvability condition.

$h^0$:

$$
-\frac{\partial^2 u_2}{\partial \eta^2} - \lambda u_2 = -\kappa \frac{\partial}{\partial \eta} u_1 - \left( \kappa^2 \eta \frac{\partial}{\partial \eta} - \frac{\partial^2}{\partial s^2} \right) u_0 + \lambda_0 u_0 = 
$$

$$
= -\varphi_1'\kappa^2 v_0 - \varphi_0'\kappa w_0 - \eta \varphi_0'\kappa^2 v_0 + \varphi_0 v_0'' + \lambda_0 \varphi_0 v_0.
$$

As earlier, the solvability condition for this equation consists in the orthogonality of the right hand side to $\varphi_0$ for any $s$,

$$
\int_0^1 (-\varphi_1'\kappa^2 v_0 - \varphi_0'\kappa w_0 - \eta \varphi_0'\kappa^2 v_0 + \varphi_0 v_0'' + \lambda_0 \varphi_0 v_0) \varphi_0 \, d\eta = 0.
$$

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One can easily check that
\[
\int_0^1 \varphi'_0 \varphi_0 d\eta = 0, \\
\int_0^1 \eta \varphi'_0 \varphi_0 d\eta = -\frac{1}{2} \int_0^1 \varphi_0^2 d\eta \\
\int_0^1 \varphi'_1 \varphi_0 d\eta = \frac{1}{4} \int_0^1 \varphi_0^2 d\eta.
\]
(3.15)

Hence from (3.14) and (3.15) we obtain the equation for \( v_0 \):
\[
- v''_0 - \frac{1}{4} \kappa^2 v_0 = \lambda_0 v_0, \quad s \in (0, 1).
\]
(3.16)

From the setting of the original problem it follows naturally that the Dirichlet boundary condition has to be prescribed at the right end of the interval,
\[ v_0(1) = 0, \]
but the condition at the left end is still to be determined. Substituting equation (3.16) back into (3.13) we transform it into
\[
- \frac{\partial^2 u_2}{\partial \eta^2} - \lambda u_2 = \left( -\frac{3}{4} \varphi_0 - \frac{3}{2} \eta \varphi'_0 \right) \kappa^2 v_0 - \varphi'_0 \kappa w_0.
\]
(3.17)

A solution to this equation is given by the formula
\[
u_2 = \varphi_2 v_2 + \varphi_1 w_1 + \varphi_0 z_0, \\
v_2 = \kappa^2 v_0, \\
\varphi_2 = \frac{3}{8} \eta^2 \varphi_0, \\
w_1 = \kappa w_0,
\]
(3.18)

where \( z_0(s) \) is an arbitrary function. For the purposes of the present chapter we choose it to be identically zero,
\[ z_0 \equiv 0, \quad s \in [0, 1]. \]
In the next step we obtain an equation for $w_0$.

\[
\frac{\partial^2 u_3}{\partial \eta^2} - \lambda_2 u_3 = -\kappa \frac{\partial u_2}{\partial \eta} - \frac{\kappa^2}{\eta} \frac{\partial u_1}{\partial \eta} + \frac{\partial^2 u_1}{\partial s^2} - \kappa^3 \frac{\partial u_0}{\partial \eta} + 2\kappa \eta \frac{\partial^2 u_0}{\partial s^2} + \kappa \eta \frac{\partial u_0}{\partial s} + \lambda_0 u_1 + \lambda_1 u_0. \tag{3.19}
\]

As usual, the right hand side must be orthogonal to $\varphi_0$. So we multiply the right hand side by $\varphi_0$, integrate over the interval $[0, 1]$ with respect to $\eta$ and work out all the terms separately. (Note that $\int_0^1 \eta \varphi_0^2 d\eta = \frac{1}{2} \int_0^1 \varphi_0^2 d\eta$ and $\int_0^1 \eta^2 \varphi_0^2 d\eta = \int_0^1 \varphi_0^2 d\eta$.) As a result,

\[
\begin{align*}
- \int_0^1 \kappa \frac{\partial u_2}{\partial \eta} \varphi_0 d\eta &= - \left( \frac{3}{16} \kappa^3 v_0 + \frac{1}{4} \kappa^2 w_0 \right) \int_0^1 \varphi_0^2 d\eta, \\
- \int_0^1 \kappa^2 \eta \frac{\partial u_1}{\partial \eta} \varphi_0 d\eta &= \frac{1}{2} \kappa^2 w_0 \int_0^1 \varphi_0^2 d\eta, \\
\int_0^1 \frac{\partial^2 u_1}{\partial s^2} \varphi_0 d\eta &= \left( \frac{1}{4} (\kappa'' v_0 + 2\kappa' v_0' + \kappa v_0'' + w_0'') \right) \int_0^1 \varphi_0^2 d\eta, \\
- \int_0^1 \kappa^3 n^2 \kappa \frac{\partial u_0}{\partial \eta} \varphi_0 d\eta &= \frac{1}{2} \kappa^3 v_0 \int_0^1 \varphi_0^2 d\eta, \\
\int_0^1 2\kappa n \frac{\partial^2 u_0}{\partial s^2} \varphi_0 d\eta &= \kappa v_0'' \int_0^1 \varphi_0^2 d\eta, \\
\int_0^1 \kappa' n \frac{\partial u_0}{\partial s} \varphi_0 d\eta &= \frac{1}{2} \kappa' v_0' \int_0^1 \varphi_0^2 d\eta, \\
\int_0^1 \lambda_0 u_1 \varphi_0 d\eta &= \left( \frac{1}{4} \lambda_0 \kappa v_0 + \lambda_0 w_0 \right) \int_0^1 \varphi_0^2 d\eta, \\
\int_0^1 \lambda_1 u_0 \varphi_0 d\eta &= \lambda_1 v_0 \int_0^1 \varphi_0^2 d\eta.
\end{align*}
\]
We take the sum of the expressions on the right hand side of the above, equate it to zero and use equation (3.16) to eliminate the second derivatives of $v_0$. Thus the solvability condition gives us the following equation for $w_0$:

$$-w''_0 - \frac{1}{4}\kappa^2 w_0 - \lambda_0 w_0 = \lambda_1 v_0 - \lambda_0 \kappa v_0 + \frac{1}{4}\kappa'' v_0 + \kappa' v'_0, \ s \in (0, 1). \quad (3.20)$$

The solution to (3.19) (if it exists) is not unique. We fix a particular one by imposing the orthogonality condition:

$$\int_{0}^{1} u_{3}(s, \eta) \varphi_0(\eta) d\eta \equiv 0. \quad (3.21)$$

So, in summary, the formal asymptotic approximation to the solution of (3.2) is given by

$$u_{h}^{(3)} = \sum_{i=0}^{3} h^{i} u_{i}(s, \eta), \quad \lambda_{h}^{(3)} = h^{-2} \lambda_{-2} + \lambda_0 + h\lambda_1. \quad (3.22)$$

The eigenvalues $\lambda_0$ and $v_0$ (as well as $\lambda_1$ and $w_0$) are not yet defined, since the boundary condition at the left end of the interval $(0, 1)$ is unclear. In order to obtain proper boundary conditions on $v_0$ and $w_0$ we need to match the asymptotics (3.22) with the asymptotics of a solution of the inner problem, i.e. a solution near the origin which satisfy Neumann boundary condition on $\gamma_1$.

### 3.3 Inner problem and scattering matrix

In order to obtain proper boundary conditions for functions $v_0$ we need to consider the behaviour of the solution of (3.2) in the neighbourhood of the origin. We will use the method of matched asymptotic expansions, adjusting expansion (3.22) to the asymptotic expansion of the solution to problem (3.2) near the origin. From now on we assume that $\lambda_0 \neq 0$. By the assumptions of this chapter the domain $\Omega_h$ in the neighbourhood of the origin coincides with a straight strip of the width $h$ slanted at the origin. Then we can introduce a stretched variable $y = h^{-1}x$ (hence $\Delta_x = h^{-2} \Delta_y$) and consider an ‘inner’ eigenvalue problem in a semi-infinite cylinder (see Figure 3-2)

$$\Pi_\alpha := \{y \mid y_1 > 0, y_2 \in (0, 1), y_2 < \tan(\alpha) y_1\}, \quad (3.23)$$
Figure 3-2: Semi-infinite cylinder

\[-\Delta_y g(y) = k_h^2 g(y), \quad y \in \Pi_\alpha,\]
\[\frac{\partial g}{\partial \nu} = 0, \quad y \in \Gamma_\alpha^1,\]
\[g = 0, \quad y \in \Gamma_\alpha^2,\]
\[(\nu \text{ is the exterior normal in the } y\text{-coordinates})\]

\[k_h^2 = (\pi^2 + h^2\lambda_0 + h^3\lambda_1),\]  

via (3.9) and (3.11). Here \(\Gamma_\alpha^1\) is the slanted part of the boundary of \(\Pi_\alpha\) (i.e. corresponding to \(y_2 = \tan(\alpha) y_1\)) and \(\Gamma_\alpha^2 = \partial \Pi_\alpha \backslash \Gamma_\alpha^1\). Denote \(\mu_h = \sqrt{\lambda_0 + h\lambda_1}\).

A solution to problem (3.24) depends obviously on the angle \(\alpha\). We therefore use index \(\alpha\) in our notation wherever necessary.

In this section we will make some use of the reasoning and results of [66]. In general, equation (3.24) does not have a nontrivial solution from \(L^2(\Pi_\alpha)\). Nevertheless, there always exists a solution that is given us a sum of the Floquet waves and of some function decaying exponentially at infinity. Its structure depends on the values of \(\alpha\) and \(k_h\). The term ‘Floquet waves’ is used here for the solutions of the eigenvalue problem akin to (3.24) where the domain \(\Pi_\alpha\) is replaced by the infinite strip \(0 < y_2 < 1\). These solutions are given by the following formula

\[\psi_{\pm}(y) = \exp(\pm ih\mu_h y_1) \sin(\pi y_2) \text{ if } \mu_h \neq 0.\]  

Formally setting \(h = 0\) we have \(k_0 = \pi^2\). We call this threshold case. The Floquet waves then are the following

\[\psi_0(y) = \sin(\pi y_2), \quad \psi_1(y) = y_1 \sin(\pi y_2).\]  

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When $\lambda_0$ is positive (super-threshold case) the Floquet waves are oscillating waves of constant amplitude. They have clear physical meaning. Namely

$$g^+ := \exp(-i\hbar\mu_1 y_1) \sin(\pi y_2), \quad (3.28)$$

is called the incoming wave (as travelling from plus infinity), and

$$g^- := \exp(i\hbar\mu_1 y_1) \sin(\pi y_2), \quad (3.29)$$

is called the outgoing wave (as travelling to plus infinity). When $\lambda_0$ is negative (sub-threshold case) the Floquet waves are exponentially growing and exponentially decaying functions. In this case there is no similar intuitive classification. Nevertheless, for some technical reasons (see [66] for some explanations), it is convenient to call the following combinations of Floquet waves the incoming and the outgoing waves respectively:

$$g^+ := \frac{1}{\sqrt{2}} [\exp(i\hbar\mu_1 y_1) - i \exp(-i\hbar\mu_1 y_1)] \sin(\pi y_2),$$

$$g^- := \frac{1}{\sqrt{2}} [\exp(i\hbar\mu_1 y_1) + i \exp(-i\hbar\mu_1 y_1)] \sin(\pi y_2). \quad (3.30)$$

(Notice that the normalising coefficient $\frac{1}{\sqrt{2}}$ is introduced to make the amplitude of the waves the same as in (3.28), (3.29).)

Problem (3.24) is solvable in weighted spaces, see e.g. [66], and the solution could be written as a linear combination of incoming and outgoing waves $g^+$ and $g^-$ and some exponentially decaying function $z$,

$$g = g^+ + Sg^- + z. \quad (3.31)$$

The function $z$ decays exponentially in the following sense. Let $H^2_\beta(\Pi_\alpha)$ be a completion in the norm $\|e^{\beta y_1 u}\|_{H^2(\Pi_\alpha)}$ of the set of functions in $C^\infty(\Pi_\alpha)$ with compact supports vanishing in the neighbourhood of $\Gamma^2_\alpha$. Then we require $z \in H^2_\beta(\Pi_\alpha)$ for some positive $\beta$. Finally, $S$ is a unitary scattering matrix (in the present case it is simply a complex number depending on $h$ and $\mu_1$, $|S| = 1$), whose asymptotic behaviour as $h \to 0$ determines in fact the boundary conditions for equations (3.16), (3.20). One can write the asymptotics of $S$ in terms of the scattering matrix $s$ relevant to Floquet waves (3.27) on the threshold, as follows.
Solutions \( \sin(\pi y_2) \) and \( y_1 \sin(\pi y_2) \) correspond to the spectral parameter \( k_0^2 = \lambda_{-2} = \pi^2 \), i.e. by formally setting \( h = 0 \) in (3.25). The incoming and outgoing waves in this case will be defined as

\[
\phi^+ \coloneqq \frac{1}{\sqrt{2}} (1 - iy_1) \sin(\pi y_2), \\
\phi^- \coloneqq \frac{1}{\sqrt{2}} (1 + iy_1) \sin(\pi y_2).
\]  

(3.32)

We assume that there does not exist a solution of (3.24) with \( k_0^2 = \pi^2 \) belonging to \( L^2(\Pi_\alpha) \). Then the solution to the problem in a weighted space can be presented in the form

\[
\phi = \phi^+ + \phi^- + z,
\]  

(3.33)

where \( \phi \), \( |\phi| = 1 \), is a scattering matrix and \( z \in H^2(\Pi_\alpha) \) for some \( \beta > 0 \) (this \( z \) is obviously different from the one in (3.31)).

The asymptotics of \( S \) can then be written in terms of \( \phi \) as follows. For \( \phi \neq 1 \) one can obtain an explicit formula for the second term of the asymptotics of \( S \), see [66]. Namely, when \( \lambda_{0} \) is negative

\[
S = i - h2\mu_h \frac{1 + \phi}{1 - \phi} + O(h^2) = i - h2\sqrt{\lambda_{0}} \frac{1 + \phi}{1 - \phi} + O(h^2);
\]  

(3.34)

and when \( \lambda_{0} \) is positive

\[
S = -1 + h2\mu_h \frac{1 + \phi}{1 - \phi} + O(h^2) = -1 + h2\sqrt{\lambda_{0}} \frac{1 + \phi}{1 - \phi} + O(h^2).
\]  

(3.35)

(In our notation \( \sqrt{\lambda_{0}} = i\sqrt{|\lambda_{0}|} \) for \( \lambda_{0} < 0 \).)

It is obvious, that in the case \( \phi = 1 \) (the critical case) formulas (3.34), (3.35) are unsuitable. We will derive an asymptotics of the scattering matrix

\[
S = \sum_{m=0}^{\infty} h^m S_m
\]  

(3.36)

when \( \phi = 1 \) also following the general reasoning from [66]. (Notice in passing that the following derivation is easily adopted to the simpler non-critical cases yielding (3.34) and (3.35).) The special solution of (3.24) is given by (3.31).
We seek the asymptotics of the solution in the form

\[ g \approx g^+ + \sum_{m=0}^{\infty} h^m S_m g^- \quad \text{for} \quad y_1 > \log h, \]  

(3.37)

\[ g \approx \sum_{m=0}^{\infty} h^m V_m(y) \quad \text{for} \quad y_1 < 2 \log h. \]  

(3.38)

Matching these two expansions in the intermediate region \( |\log h| < y_1 < 2 |\log h| \) we can find (3.36).

Substituting (3.38) into (3.24) we obtain the following sequence of boundary value problems

\[-(\Delta + k_0^2) V_m = 0, \quad y \in \Pi_\alpha, \quad m = 0, 1,\]
\[-(\Delta + k_0^2) V_m = \mu_2^2 V_{m-2}, \quad y \in \Pi_\alpha, \quad m = 2, 3, \ldots,\]
\[\frac{\partial V_m}{\partial \nu} = 0, \quad y \in \Gamma_1^1,\]
\[V_m = 0, \quad y \in \Gamma_2^0, \quad m = 0, 1, \ldots.\]  

(3.39)

For \( m = 0, 1 \) the solution is given by

\[ V_m = A_m \tilde{g}, \]

where \( \tilde{g} \) is from (3.33). Then for \( m = 2 \) we obtain

\[-(\Delta + k_0^2) V_2 = A_0 \mu_2^2 (\tilde{g}^+ + \tilde{g}^- + z)\]

(recall that \( \tilde{s} = 1 \)). A solution to this problem exists and has the form

\[ V_2 = -\frac{1}{2} A_0 \mu_2^2 y_1^2 (\tilde{g}^+ + \tilde{g}^-) + \tilde{V}_2. \]  

(3.40)

The function \( \tilde{V}_2 \) solves the following boundary value problem

\[-(\Delta + k_0^2) \tilde{V}_2 = z,\]
\[\frac{\partial \tilde{V}_2}{\partial \nu} = \frac{\partial}{\partial \nu} \left( \frac{1}{2} A_0 \mu_2^2 y_1^2 (\tilde{g}^+ + \tilde{g}^-) \right), \quad y \in \Gamma_1^1,\]
\[\tilde{V}_2 = 0, \quad y \in \Gamma_2^0.\]
\( \tilde{V}_2 \) is given by
\[
\tilde{V}_2 = A_2 \hat{g} + B \hat{g} - z \tag{3.41}
\]
where \( A_2 \) and \( B \) are some constants and \( z \in H^2 \), see [60].

Let us derive a formula for the coefficient \( B \). We need this because, as we will see later, \( B \) enters in the formula for the first order (\( O(h) \)) term in the asymptotics of \( S \). In order to obtain the formula we apply integration by parts to the following integral (bar over a symbol denotes its complex conjugate).
\[
0 = \int_{\Pi_{\alpha,R}} (\Delta + k_0^2) \hat{g} V^2 dy = \int_{\Pi_{\alpha,R}} \hat{g} (\Delta + k_0^2) V^2 dy +
+ \int_{\partial \Pi_{\alpha,R}} \frac{\partial}{\partial \nu} \hat{g} V^2 dS - \int_{\partial \Pi_{\alpha,R}} \hat{g} \frac{\partial}{\partial \nu} V^2 dS = \tag{3.42}
\]
\[
= -A_0 \mu_h^2 \int_{\Pi_{\alpha,R}} |\hat{g}|^2 dy + \int_{\partial \Pi_{\alpha,R}} \frac{\partial}{\partial \nu} \hat{g} V^2 dS - \int_{\partial \Pi_{\alpha,R}} \hat{g} \frac{\partial}{\partial \nu} V^2 dS,
\]
where \( \Pi_{\alpha,R} \) denotes the part of \( \Pi_\alpha \) satisfying condition \( y_1 < R \). It becomes clear from the last formula why we integrate over the bounded domain: the reason is that function \( \hat{g} \) does not belong to \( L^2(\Pi_\alpha) \). Due to the boundary conditions and asymptotic behaviour of \( \hat{g} \) and \( V^2 \) the second term on the right hand side of the latter converges to zero as \( R \to \infty \). As for the last term, we derive via (3.32), (3.33), (3.40) and (3.41) the following
\[
\int_{\partial \Pi_{\alpha,R}} \hat{g} \frac{\partial}{\partial \nu} V^2 dS = \int_{y_1=R} \hat{g} \frac{\partial}{\partial y_1} V^2 dy_2 =
\]
\[
= - \int_{y_1=R} \sin(\pi y_2) (2A_0 \mu_h^2 y_1 \sin(\pi y_2) + iB \sin(\pi y_2)) dy_2 + o(1) = \tag{3.43}
\]
\[
= -A_0 \mu_h^2 R - \frac{i}{2} B + o(1).
\]

From (3.42) and (3.43) we obtain
\[
B = -i2A_0 \mu_h^2 \lim_{R \to \infty} \left[ \int_{\Pi_{\alpha,R}} |\hat{g}|^2 dy - R \right]. \tag{3.44}
\]
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Notice that the limit in the latter formula indeed exists. This is easy to see from the following observation:

\[ R = R \int_{y_1 = R}^{\infty} |\tilde{g}^+ + \tilde{g}^-|^2 dy_2 = \int_{\Pi_0, R} |\tilde{g}^+ + \tilde{g}^-|^2 dy + \text{const.} \]

Let us denote

\[ \sigma = \lim_{R \to \infty} \left[ \int_{\Pi_0, R} |\tilde{g}|^2 dy - R \right]. \tag{3.45} \]

Then

\[ B = -i2A_0\mu_h^2 \sigma. \]

Let us write the asymptotics for the Floquet waves as \( h \to 0 \) and \( y_1 \sim |\log h| \):

\[ \exp(\pm ih\mu_h y_1) = 1 \pm i\mu_h y_1 - \frac{1}{2} h^2 \mu_h^2 y_1^2 + O(h^3 |\log h|^3). \]

Notice that

\[ \frac{1}{2} (\tilde{g}^+ + \tilde{g}^-) = \frac{1}{\sqrt{2}} \sin(\pi y_2), \]

\[ \frac{1}{2} (\tilde{g}^- - \tilde{g}^+) = i \frac{1}{\sqrt{2}} y_1 \sin(\pi y_2). \]

First we consider case of negative \( \lambda_0 \). Then we can write asymptotics (3.37) as follows,

\[ g = \sigma_- (\tilde{g}^+ + \tilde{g}^-) + S_0 \sigma_+ (\tilde{g}^+ + \tilde{g}^-) + h[\sigma_+ \mu_h (\tilde{g}^- - \tilde{g}^+) + \]

\[ + S_0 \sigma_- \mu_h (\tilde{g}^- - \tilde{g}^+) + S_1 \sigma_+ (\tilde{g}^+ + \tilde{g}^-)] + h^2 \left[ -\frac{1}{2} \sigma_- \mu_h^2 y_1^2 (\tilde{g}^+ + \tilde{g}^-) - \frac{1}{2} S_0 \sigma_+ \mu_h^2 y_1^2 (\tilde{g}^- + \tilde{g}^+) + \right. \]

\[ - S_1 \sigma_- \mu_h (\tilde{g}^- - \tilde{g}^+) + S_2 \sigma_+ (\tilde{g}^+ + \tilde{g}^-)] + \]

\[ + O(h^3 |\log h|^3), \tag{3.46} \]

where we denote

\[ \sigma_\pm = \frac{1 \pm i}{2}. \]

Now we derive the first two terms of asymptotics (3.36). Matching expansions (3.37) and (3.38) in the intermediate region \( |\log h| < y_1 < 2 |\log h| \) we first equate
main terms of $V_0$ and term of order one in $(3.46)$. From this we obtain

$$A_0 = \sigma_+ (-i + S_0).$$

Equating the terms of order $h$ and collecting the coefficients at $\hat{g}^+$ and $\hat{g}^-$ we respectively have two equations

$$A_1 = \sigma_+ (-\mu_h + i \mu_h S_0 + S_1)$$

and

$$A_1 = \sigma_+ (\mu_h - i \mu_h S_0 + S_1).$$

Hence it follows that

$$A_1 = S_1,$$

$$S_0 = -i,$$

$$A_0 = 2\sigma_-.$$

Equating terms of order $h^2$ we obtain

$$A_2 = \sigma_+ (i \mu_h S_1 + S_2)$$

and

$$A_2 + B = \sigma_+ (-i \mu_h S_1 + S_2).$$

Then we arrive at

$$S_1 = \frac{\sigma_+}{\mu_h} B = -i2\mu_h \sigma.$$

Notice that since $\mu_h$ is purely imaginary the first order corrector $S_1$ is real. So, the asymptotics of $S$ in case when $\hat{s} = 1$ and $\lambda_0$ is negative is given by

$$S = -i - hi2\mu_h \sigma + O(h^2) = -i - hi2\sqrt{\lambda_0} \sigma + O(h^2). \quad (3.47)$$

Analogously we obtain the asymptotics for the case of positive $\lambda_0$. Now, the asymptotics of $g$ is given by

$$g = \frac{1}{\sqrt{2}} (1 + S_0)(\hat{g}^+ + \hat{g}^-) + h[\frac{1}{\sqrt{2}} \mu_h (\hat{g}^- - \hat{g}^+) + \frac{1}{\sqrt{2}} S_0 \mu_h (\hat{g}^+ - \hat{g}^-) +$$

$$+ \frac{1}{\sqrt{2}} S_1 (\hat{g}^+ + \hat{g}^-)] + h^2 [\frac{1}{2\sqrt{2}} (1 + S_0) \mu_h y_1^2 (\hat{g}^+ + \hat{g}^-) +$$

$$+ \frac{1}{\sqrt{2}} S_1 \mu_h (\hat{g}^- - \hat{g}^+) + \frac{1}{\sqrt{2}} S_2 (\hat{g}^+ + \hat{g}^-)] + O(h^3 |\log h|^3). \quad (3.48)$$
Equating terms of the same order in (3.38) and (3.48) we derive sequentially

\[ A_0 = \frac{1}{\sqrt{2}}(1 + S_0), \]
\[ A_1 = \frac{1}{\sqrt{2}}(\mu_h - S_0\mu_h + S_1) = \frac{1}{\sqrt{2}}(-\mu_h + S_0\mu_h + S_1), \]

hence

\[ S_0 = 1, \]
\[ A_2 = \frac{1}{\sqrt{2}}(-\mu_hS_1 + S_2), \]
\[ A_2 + B = \frac{1}{\sqrt{2}}(\mu_hS_1 + S_2), \]

hence

\[ S_1 = -i2\mu_h\sigma. \]

In this case \( \mu_h \) is real and \( S_1 \) is purely imaginary. Finally we have

\[ S = 1 - hi2\mu_h\sigma + O(h^2) = 1 - hi2\sqrt{\lambda_0}\sigma + O(h^2). \]  
(3.49)

**Remark 3.3.1.** The scattering matrix \( S \) depends on the choice of a coordinate system, i.e. on the position of the domain \( \Pi_\alpha \) in a coordinate system. In particular, the formulas for \( S \) in this section are valid only for \( \Pi_\alpha \) positioned as described in (3.23).

### 3.4 Matching of asymptotics and limit boundary conditions

In this section we will derive proper boundary condition for the function \( v_0 \) at the left end of the interval \([0,1]\). In order to do this we need to match the outer asymptotic solution (3.22) to problem (3.7) and asymptotics of the solution to inner problem (3.24) given by (3.31), (3.36). The matching will be made in some intermediate region lying near \( s = 0 \). Accomplishing this we will eliminate the uncertainty about the approximate solution \( u_h^{(3)} \) in (3.22).

Note that due to the straight shape of \( \Omega_h \) when \( s < s_0 \) the coordinates are related by the formula \( (s, \eta) = (hy_1, y_2) \). In this section we mostly use the coordinates \( (s, \eta) \), so we must rewrite formulas from the previous section, in
particular the Floquet waves (3.26) read

$$\psi^\pm = \exp(\pm i\mu_h s) \sin(\pi\eta) = \exp(\pm i\mu_h s) \varphi_0(\eta).$$

We carry out the matching of the asymptotic expansions in the region $s \in (h^{1/3}, 2h^{1/3})$. For such $s$ the curvature $\kappa$ is identically zero. Then it is easy to see that equation (3.19) locally becomes homogeneous, and due to condition (3.21) we have $u_3 = 0$. Also obviously we have $u_1 = \varphi_0 w_0$ and $u_2 = 0$. Thus the approximate solution to (3.2) for $s < s_0$ simplifies to

$$u_h^{(3)} = \varphi_0(\eta)(v_0(s) + hw_0(s)), \quad (3.50)$$

where $v_0$ and $w_0$ locally satisfy differential equations

$$-v''_0 = \lambda_0 v_0 \quad (3.51)$$

and

$$-w''_0 - \lambda_0 w_0 = \lambda_1 v_0,$$

cf. (3.16) and (3.20). Then $v_0$ is a linear combination of exponents,

$$v_0 = C_1 \exp(i\sqrt{\lambda_0} s) + C_2 \exp(-i\sqrt{\lambda_0} s), \quad (3.52)$$

and $w_0$, consequently, can be presented in the form

$$w_0 = C_1 \exp(i\sqrt{\lambda_0} s) \frac{i\lambda_1}{2\sqrt{\lambda_0}} s - C_2 \exp(-i\sqrt{\lambda_0} s) \frac{i\lambda_1}{2\sqrt{\lambda_0}} s +$$

$$+ C_3 \exp(i\sqrt{\lambda_0} s) + C_4 \exp(-i\sqrt{\lambda_0} s). \quad (3.53)$$

Let us write an asymptotics of the Floquet wave as $h \to 0$. Notice that

$$\mu_h = \sqrt{\lambda_0} + h\frac{\lambda_1}{2\sqrt{\lambda_0}} + O(h^2).$$

Then

$$\psi^\pm = \exp(\pm i\mu_h s) \sin(\pi\eta) =$$

$$= \exp(\pm i\sqrt{\lambda_0} s) \exp \left( \pm h \frac{i\lambda_1}{2\sqrt{\lambda_0}} s + O(h^2) \right) \varphi_0(\eta) =$$

$$= \exp(\pm i\sqrt{\lambda_0} s) \left( 1 \pm h \frac{i\lambda_1}{2\sqrt{\lambda_0}} s \right) \varphi_0(\eta) + O(h^2). \quad (3.54)$$
Consider first the non-critical case and negative $\lambda_0$. Let us denote

$$\sigma = -i \frac{1 + \overset{\circ}{s}}{1 - \overset{\circ}{s}}. \tag{3.55}$$

Notice that we use the same notation for the different objects, see (3.45) and (3.55). We do this because these objects play a similar role in the formulas for the asymptotics of the scattering matrix. Also there should not be any confusion since formula (3.45) is only used for the critical case $\overset{\circ}{s} = 1$ and (3.55) is valid for the non-critical case $\overset{\circ}{s} \neq 1$. Notice further that straightforward calculation shows that $\sigma$ is real,

$$\sigma = \frac{\text{Im} \overset{\circ}{s}}{1 - \text{Re} \overset{\circ}{s}}. \tag{3.56}$$

In view of (3.54), employing (3.34) we obtain the following asymptotics for the solution of (3.24):

$$g = \frac{1}{\sqrt{2}} \left( \exp(i \mu_h s) - i \exp(-i \mu_h s) \right) + \nonumber$$

$$+ S \left[ \exp(i \mu_h s) + i \exp(-i \mu_h s) \right] \varphi_0(\eta) + z = \nonumber$$

$$= \frac{1 + i}{\sqrt{2}} \exp(i \sqrt{\lambda_0 s}) \varphi_0(\eta) - \frac{1 + i}{\sqrt{2}} \exp(-i \sqrt{\lambda_0 s}) \varphi_0(\eta) + \nonumber$$

$$+ \frac{h}{\sqrt{2}} \left( (1 + i) \frac{i \lambda_1}{2 \sqrt{\lambda_0}} s - i 2 \sqrt{\lambda_0} \sigma \right) \exp(i \sqrt{\lambda_0 s}) \varphi_0(\eta) + \nonumber$$

$$+ \frac{h}{\sqrt{2}} \left( (1 + i) \frac{i \lambda_1}{2 \sqrt{\lambda_0}} s + 2 \sqrt{\lambda_0} \sigma \right) \exp(-i \sqrt{\lambda_0 s}) \varphi_0(\eta) + O(h^2) + z. \tag{3.56}$$

In formulas (3.54) and (3.56) notation $O(h^2)$ stands for the remainder that is an infinitely smooth and uniformly bounded with respect to $h$ function times $h^2$, i.e. this remainder together with all its derivatives is bounded by $Ch^2$ in the $L^\infty$-norm. (Constant $C$ is independent of $h$ although may depend on the order of the derivative). Another remainder is exponentially decaying: $z = z(y) \in H^1_{\beta}(\Pi_{\alpha})$. Since $z = z(h^{-1}s, \eta)$, it must be a rapidly decaying function even for relatively small values of $s$. However it depends on $h$ as a parameter, since $z$ enters formula (3.31) for the solution of eigenvalue problem (3.24) where the eigenvalue depends on $h$. Fortunately, $z(y)$ is bounded in $H^1_{\beta}(\Pi_{\alpha})$ uniformly in $h$, see 88
This means that the $H^1$-norm of $z(h^{-1}s, \eta)$ in the region $s \in (h^{1/3}, 2h^{1/3})$ is exponentially small ($\sim \exp(-h^{-2/3}/\beta)$) uniformly with respect to $h$.

We match asymptotics of $u_h^{(3)}$ and $Mg$, where $M$ is arbitrary constant. This gives us relations between coefficients in (3.52) and (3.53). Matching the main terms of the asymptotics we derive from (3.53) and (3.56) that

$$C_2 = -C_1 = -\frac{1 + i}{\sqrt{2}} M.$$  \hspace{1cm} (3.57)

It is obvious from (3.52) and (3.57) that $v_0(0) = 0$. Thus, $v_0$ and $\lambda_0$ is a solution to eigenvalue problem (3.16) with Dirichlet boundary conditions

$$v_0(0) = v_0(1) = 0.$$  \hspace{1cm} (3.58)

We assume also that $v_0$ is normalised,

$$\int_0^1 |v_0|^2 ds = 1.$$  \hspace{1cm} (3.59)

This condition fixes some precise value of the coefficients $C_1$ and $M$. Matching $w_0\varphi_0$ with the coefficient next to $h$ in the asymptotics of $Mg$ we determine the coefficients in (3.53). We see that $C_3$ and $C_4$ must be the following

$$C_3 = -i\sqrt{2\lambda_0 \sigma} M = -(1 + i)\sqrt{\lambda_0 \sigma} C_1,$$

$$C_4 = \sqrt{2\lambda_0 \sigma} M = (1 - i)\sqrt{\lambda_0 \sigma} C_1.$$  \hspace{1cm} (3.60)

It follows from (3.53) and the latter that $w_0$ must satisfy the following heterogeneous the Dirichlet condition at the point $s = 0$:

$$w_0(0) = C_3 + C_4 = -2i \sqrt{\lambda_0 \sigma} C_1.$$  \hspace{1cm} (3.61)

In order that the function $u_1 = \varphi_1 v_1 + \varphi_0 v_0$ comply with Dirichlet condition on the right end of the strip $\Omega_h$ we must set

$$w_0(1) = 0.$$  \hspace{1cm} (3.62)

Now we are going to demonstrate that there is a unique choice of $\lambda_1$ (which was not defined yet) such that there exists a solution of (3.20) satisfying boundary conditions (3.61), (3.62). Indeed, it is well known that the aforementioned prob-
lem has a solution if and only if the right hand side of (3.20) satisfies the following solvability condition:

\[
\int_0^1 \left( \lambda_1 v_0 - \lambda_0 \kappa v_0 + \frac{1}{4} \kappa'' v_0 + \kappa' v'_0 \right) \bar{v}_0 ds = -w_0(0)\bar{\sigma}_0(0). \tag{3.63}
\]

From (3.52) and (3.57) we have \( v'_0(0) = 2i\sqrt{\lambda_0}C_1 \). Then from (3.61) we obtain that

\[
-w_0\bar{\sigma}_0(0) = 4|\lambda_0||C_1|^2\sigma = |v'_0(0)|^2\sigma,
\]

since

\[
|v'_0(0)|^2 = 4|\lambda_0||C_1|^2.
\]

Notice also that in the neighbourhood of zero where \( \kappa = 0 \) the eigenfunction \( v_0 \) has a form \( C_1 \left[ \exp(-\sqrt{|\lambda_0|} s) - \exp(\sqrt{|\lambda_0|} s) \right] \). It follow from the theory of ordinary differential equations that \( v_0 = C_1 f(s), s \in [0,1] \), where \( f(s) \) is a real valued function. In this case \( v'_0 \bar{\sigma}_0 = \frac{1}{2}(|v_0|^2)' \) and one can apply integration by parts as follows,

\[
\int_0^1 \kappa' v'_0 \bar{v}_0 ds = -\frac{1}{2} \int_0^1 \kappa'' |v_0|^2 ds.
\]

Thus solvability condition (3.63) is fulfilled if \( \lambda_1 \) is given by:

\[
\lambda_1 = \left[ \int_0^1 \left( \lambda_0 \kappa + \frac{1}{4} \kappa'' \right) |v_0|^2 ds + |v'_0(0)|^2 \sigma \right] \|v_0\|^{-2}, \tag{3.64}
\]

where \( \|v_0\|^2 = \int_0^1 |v_0|^2 ds \).

Therefore, problem (3.20), (3.61), (3.62) has a solution (which actually is not unique). We need to chose a solution that satisfies (3.53), (3.57) and (3.60) (so that \( u_h^{(3)} \) would match with the inner solution \( Mg \)). Let us show that this is possible. We fix some arbitrary solution \( \tilde{w}_0 \) of (3.20), (3.61), (3.62). In the neighbourhood of zero it has a form

\[
\tilde{w}_0 = C_1 \exp(i\sqrt{|\lambda_0|} s) \frac{i\lambda_1}{2\sqrt{|\lambda_0|}} s - C_2 \exp(-i\sqrt{|\lambda_0|} s) \frac{i\lambda_1}{2\sqrt{|\lambda_0|}} s + \tilde{C}_3 \exp(i\sqrt{|\lambda_0|} s) + \tilde{C}_4 \exp(-i\sqrt{|\lambda_0|} s), \tag{3.65}
\]

where the coefficients \( C_1 \) and \( C_2 \) are as in (3.57), but \( \tilde{C}_3 \) and \( \tilde{C}_4 \) may differ.
from $C_3$ and $C_4$, cf. (3.51)–(3.53). From the boundary conditions for $\tilde{w}_0$ we have

$$\tilde{w}_0(0) = \tilde{C}_3 + \tilde{C}_4 = -2i\sqrt{\lambda_0}\sigma C_1.$$  

(3.66)

Notice that function $w_0 = \tilde{w}_0 + mv_0$, where $m$ is arbitrary constant, is also a solution of the concerned problem. In the neighbourhood of zero it can be written as

$$w_0 = \tilde{w}_0 + mv_0 = C_1 \exp(i\sqrt{\lambda_0}s) \frac{i\lambda_1}{2\sqrt{\lambda_0}} s - C_2 \exp(-i\sqrt{\lambda_0}s) \frac{i\lambda_1}{2\sqrt{\lambda_0}} s +$$

$$+(\tilde{C}_3 + mC_1) \exp(i\sqrt{\lambda_0}s) + (\tilde{C}_4 + mC_2) \exp(-i\sqrt{\lambda_0}s).$$

We can choose $m$ such that $\tilde{C}_3 + mC_1 = -(1+i)\sqrt{\lambda_0}\sigma C_1$. Then from (3.57), (3.66) we obtain that $\tilde{C}_4 + mC_2 = \tilde{C}_4 - mC_1 = (1-i)\sqrt{\lambda_0}\sigma C_1$. Hence for such choice of $m$ the solution

$$w_0 = \tilde{w}_0 + mv_0$$

behaves in neighbourhood of zero as described by (3.53), (3.57) and (3.60).

Resuming the above we conclude that such chosen solutions $v_0$ and $w_0$ match with the asymptotics of inner solution $Mg$ up to the term of order $h^2$ and we have the following relation in the region $s \in (h^{1/3}, 2h^{1/3})$:

$$Mg - u_h^{(3)} = h^2 f_1 + z,$$

(3.67)

where

$$|f_1|, \left| \frac{\partial}{\partial s} f_1 \right| \leq CM,$$

(3.68)

$$\|z\|_{H^1((h^{1/3}, 2h^{1/3}) \times (0,1))} \leq CM(h^m)$$

for any $m$.

For positive $\lambda_0$ in the non-critical case we have

$$g = \left( \exp(-i\mu_h s) + S \exp(i\mu_h s) \right) \varphi_0(\eta) + z =$$

$$= \left( -1 + hi2\sqrt{\lambda_0}\sigma - h \frac{i\lambda_1}{2\sqrt{\lambda_0}} s \right) \exp(i\sqrt{\lambda_0}s) \varphi_0(\eta) +$$

$$+ \left( 1 - h \frac{i\lambda_1}{2\sqrt{\lambda_0}} s \right) \exp(-i\sqrt{\lambda_0}s) \varphi_0(\eta) + O(h^2) + z.$$
In this case we set in (3.52) and (3.53)

\[ C_2 = -C_1 = M, \]
\[ C_3 = -2i \sqrt[4]{\lambda_0} \sigma C_1, \]
\[ C_4 = 0. \]

We choose \( v_0 \) being a solution of (3.16), (3.58) satisfying (3.59) and (3.52), (3.69).

Analogously to the above one can show that for \( \lambda_1 \) given by (3.64) there exists a solution of (3.20), (3.61), (3.62) satisfying (3.53), (3.69). It is easy to see then that (3.67) holds true.

In the critical case \( \overset{\circ}{s} = 1 \) we similarly obtain the coefficients \( C_i, i = 1, 2, 3, 4 \).

For the case of negative \( \lambda_0 \) we have

\[ C_1 = C_2 = \frac{1 - i}{\sqrt{2}} M, \]
\[ C_3 = (1 - i) \sqrt{\lambda_0} \sigma C_1, \]
\[ C_4 = (1 + i) \sqrt{\lambda_0} \sigma C_1, \]

and if \( \lambda_0 \) is positive, then

\[ C_1 = C_2 = M, \]
\[ C_3 = -2i \sqrt{\lambda_0} \sigma C_1, \]
\[ C_4 = 0. \]

These formulas imply that \( v_0 \) must satisfy Neumann boundary condition at zero.

Thus \( v_0 \) is a normalised solution of (3.16) subject to boundary conditions

\[ v'_0(0) = v_0(1) = 0. \]

In this case the solvability condition for the equation for \( w_0 \):

\[ \int_0^1 \left( \lambda_1 v_0 - \lambda_0 \kappa v_0 + \frac{1}{4} \kappa'' v_0 + \kappa' v'_0 \right) \overline{v_0} ds = w'_0(0) \overline{v_0}(0). \]

involves the value of \( w'_0 \) at zero. So we impose the following boundary conditions

\[ w'_0(0) = 2\lambda_0 \sigma C_1, \quad w_0(1) = 0, \]
from which the first one is implied by (3.53) and (3.70) (or (3.71)). Then
\[ w'_0(0)v_0(0) = 4\lambda_0|C_1|^2\sigma = |v_0(0)|^2\lambda_0\sigma. \] (3.74)

Consequently from (3.73) we obtain that
\[ \lambda_1 = \int_0^1 \left( \lambda_0\kappa + \frac{1}{4}\kappa'' \right)|v_0|^2ds + |v_0(0)|^2\lambda_0\sigma \|v_0\|^2. \] (3.75)

Then one can show that there exits a solution of (3.20), (3.74) which satisfies (3.53) and (3.70) (or (3.71)) and we still have (3.67).

Thus \( w_0 \) is fully defined as a solution of (3.20) with Dirichlet boundary condition at the right end of the interval and satisfying condition (3.53) with an appropriate coefficients near the left end. For such \( w_0 \) the solvability condition for equation (3.19) is fulfilled, hence there exists a solution \( u_3 \in C^\infty(D) \) such that \( u(s,0) = u(s,1) \equiv 0 \) and
\[ \int_0^1 u_3(s,\eta)\varphi_0(\eta)d\eta \equiv 0. \]

### 3.5 Error bounds and justification of the asymptotics

In this section we justify the asymptotics obtained earlier. In order to do this we first need to construct a function satisfying the boundary conditions in (3.2) such that after the substitution into equation (3.2) we get asymptotically small (of order \( h^{3/2} \)) error on its right hand side. It is well known that the operator \( A_h \) (as elliptic and defined in bounded domain) has a discrete spectrum with the only accumulation point at infinity. Let \( \lambda_{1,h} \leq \lambda_{2,h} \leq \ldots \) be all the eigenvalues of \( A_h \) repeating accordingly to there multiplicity and \( u_{i,h}, i \in \mathbb{N} \) be the corresponding orthonormalised eigenfunctions. Let us introduce a smooth cut-off function
\[ \chi(s) = \begin{cases} 1, & s \leq 1, \\ 0, & s \geq 2. \end{cases} \] (3.76)

We formulate the main results of the present section in the following theorem.
Theorem 3.5.1. Let $\lambda^{(3)}_h$ be given by (3.22) and $\lambda_0 \neq 0$. Then there exists $h_0 > 0$ and a constant $C$ independent of $h$ such that for any $0 < h \leq h_0$ there exist an eigenvalue $\lambda_{i,h}$ of the operator $A_h$ such that 

$$|\lambda_{i,h} - \lambda^{(3)}_h| \leq Ch^{3/2}. \tag{3.77}$$

Moreover, a function 

$$u^{\text{appr}}_h = \chi(sh^{-1/3})Mg + (1 - \chi(sh^{-1/3}))(u_0 + hu_1),$$

where $g$ is a solution of scattering problem (3.24) and $M$ is a constant such that (3.67) is satisfied, approximates eigenfunctions of $A_h$ in the following sense: for any $d > 0$ and any $0 < h \leq h_0$ there exist coefficients $c_i(h)$ such that 

$$\left\| u^{\text{appr}}_h - \sum_{|\lambda_{i,h} - \lambda^{(3)}_h| \leq d} c_i(h)u_{i,h} \right\|_{L^2(\Omega_h)} \leq Cd^{-1}h^2. \tag{3.78}$$

Remark 3.5.2. The error estimate in (3.78) is somewhat deceptive. The fact is that the norm of $u^{\text{appr}}_h$ is not of order one. Indeed, roughly speaking the main term of the asymptotics $u^{\text{appr}}_h$ is $v_0(s)\sin(h^{-1/2}n)$, where $\int_0^1 v_0^2 ds = 1$. It is clear then that the norm of $v_0(s)\sin(h^{-1/2}n)$ in $u^{\text{appr}}_h$ is of order $h^{1/2}$. One can consider normalised $u^{\text{appr}}_h$, for which the error estimate (3.78) holds with the right hand side equal $Cd^{-1}h^{3/2}$ (which is of the same order as the estimate for eigenvalues). But in this case the main term of $u^{\text{appr}}_h$ is of order $h^{-1/2}$ in $L^\infty$-norm. This seems to us to be improper in some way, so we prefer to normalise $v_0$ rather than $u^{\text{appr}}_h$.

Proof. We will first mention the regularity properties of the functions $g$ and $u^{(3)}_h$. Obviously $\varphi_0 = \sin(\pi n) \in C^\infty([0,1])$. From the general theory of ordinary differential equations we know that $v_0$ and consequently $w_0$ belong to $C^\infty([0,1])$. Then obviously $u_0, u_1, u_2 \in C^\infty(\overline{D})$ as elementary combinations of $C^\infty$ functions. The third term of the asymptotics $u_3 \in C^\infty(\overline{D})$ as a solution of ordinary differential equation with respect to $\eta$ (3.19), where $s$ plays the role of a parameter and the right hand side belongs to $C^\infty(\overline{D})$. Furthermore, in the coordinates $(s,\eta)$ the operator $-\Delta$ is presented in the form 

$$-\Delta = -\Delta_h + h^2 L_h,$$
where $L_h$ is a second order differential operator with smooth bounded coefficients, cf. (3.5), (3.6). Due to equations (3.10), (3.12), (3.13), (3.19) the approximation $u_h^{(3)}$ solves the following equation

$$-\Delta u_h^{(3)} = \lambda_h^{(3)} u_h^{(3)} + h^2 f_2 \text{ in } D,$$

(3.79)

where

$$f_2(s, \eta) = L_h u_h^{(3)} + \left( \kappa^2 \eta^2 \frac{\partial}{\partial \eta} - 2 \kappa \eta \frac{\partial^2}{\partial s^2} - \kappa' \eta \frac{\partial}{\partial s} \right) u_1 +
$$

$$+ \left( \kappa^2 \eta \frac{\partial}{\partial \eta} - \frac{\partial^2}{\partial s^2} \right) u_2 + \kappa \frac{\partial}{\partial \eta} u_3 -
$$

$$- h^2 \lambda_0 (u_2 + h u_3) - h^2 \lambda_1 (u_1 + h u_2 + h^2 u_3).$$

It is smooth and hence

$$|f_2| \leq C$$

(3.80)

uniformly in $h$.

In order to justify the asymptotics, the approximation to the actual solution of (3.2) must satisfy boundary conditions imposed in (3.2). To this end we will slightly modify the function $u_h^{(3)}$. Namely, since the functions $u_i$, $i = 0, 1, 2$, comply with the proper boundary conditions on $\gamma_2$, and $u_3$ vanishes everywhere on $\gamma_2$ except the part corresponding to $s = 1$, we need only to multiply $u_3$ by the appropriate cut-off function. Consider function

$$\hat{u}_h^{(3)} = \sum_{i=0}^2 h^i u_i + h^3 u_3 \chi((s - 1) h^{-\beta} + 2),$$

Where $\beta$ is some positive number. Obviously this function satisfies Dirichlet boundary condition on the whole of $\gamma_2$. We can rewrite it as

$$\hat{u}_h^{(3)} = u_h^{(3)} + h^3 u_3 \left[ \chi((s - 1) h^{-\beta} + 2) - 1 \right].$$

Let us derive the equation for the latter. We will drop the argument of the function $\chi((s - 1) h^{-\beta} + 2)$ to shorten the notation hoping that this will not lead to any confusion. From (3.79) we obtain

$$-\Delta \hat{u}_h^{(3)} = -\Delta u_h^{(3)} - h^3 \Delta [u_3 (\chi - 1)] = \lambda_h^{(3)} \hat{u}_h^{(3)} + h^2 f_2 + h^3 f_3 \text{ in } D,$$

(3.81)
where
\[ f_3 = -\Delta [u_3(\chi - 1)] - \lambda_h^{(3)} u_3(\chi - 1). \]

Due to formula (3.6) we have
\[
 f_3 = (-\Delta u_3 - \lambda_h^{(3)} u_3)(\chi - 1) + \tilde{f}_3 \frac{\partial}{\partial s} \chi + \tilde{f}_3 \frac{\partial^2}{\partial s^2} \chi =
\]
\[
 = (-\Delta u_3 - \lambda_h^{(3)} u_3)(\chi - 1) + h^{-\beta} \tilde{f}_3 \chi' + h^{-2\beta} \tilde{f}_3 \chi''.
\]

The components of the latter formula are bounded as follows.
\[
|\tilde{f}_3|, |\hat{f}_3| \leq C,
\]
uniformly with respect to \( h \). Since \( u_3 \in C^\infty(D) \) and bounded together with its derivatives uniformly in \( h \), and \( \Delta \) in the coordinates \( (s, \eta) \) is an operator of second order with smooth coefficients of order \( h^{-2} \), we have
\[
|\Delta u_3| \leq h^{-2} C,
\]
uniformly with respect to \( h \). It is important that support of the functions \( \chi - 1, \chi' \) and \( \chi'' \) defined on the interval \([0, 1]\) is small, namely, \( \sup(\chi - 1) = \sup(\chi') = \sup(\chi'') = [1 - h^{\beta}, 1] \). Then we arrive at
\[
|f_3| \leq (h^{-2} + h^{-2\beta})C,
\]
\[
\sup(f_3) = [1 - h^{\beta}, 1]. \tag{3.82}
\]

In order to comply with the Neumann boundary condition on \( \gamma^1 \) we replace \( \tilde{u}_h^{(3)}(s, h\eta) \) by \( g(hs, h\eta) \) in the small neighbourhood of the origin, where \( g(y_1, y_2) \) is the matching inner solution (3.31). In this neighbourhood \( \kappa \equiv 0 \) and due to (3.24) we have
\[
-\Delta g = \lambda_h^{(3)} g, \tag{3.83}
\]
and \( g \) satisfies conditions
\[
\frac{\partial g}{\partial \nu} = 0 \text{ on } \gamma^1,
\]
\[
g = 0 \text{ on } \gamma^2.
\]
Since on the interval \( s \in [h^{1/3}, 2h^{1/3}] \) we have relation (3.67), we match there the outer and inner solutions \( \tilde{u}_h^{(3)} \) and \( Mg \).

We choose the following function as an approximate solution to the eigenvalue
problem (3.2),
\[ \hat{u}_h^{\text{appr}}(s, \eta) = \chi(s h^{-1/3}) M g(h s, \eta) + (1 - \chi(s h^{-1/3})) \hat{u}_h^{(3)}(s, \eta). \]

Notice that in the region \( s \in [h^{1/3}, 2h^{1/3}] \), since the curvature \( \kappa \) is zero and the corresponding part of the domain \( \Omega_h \) is a strip parallel to the axis \( x \), the operator simply has form
\[ -\Delta = -\frac{\partial^2}{\partial s^2} - h^{-2} \frac{\partial^2}{\partial \eta^2}. \]

Then we derive via (3.83), (3.81) and (3.67)
\[ -\Delta \hat{u}_h^{\text{appr}} = -\chi M \Delta g - (1 - \chi) \Delta \hat{u}_h^{(3)} - 2 \frac{\partial}{\partial s} \frac{\partial}{\partial s} (M g - \hat{u}_h^{(3)}) - \frac{\partial^2}{\partial s^2} \chi(M g - \hat{u}_h^{(3)}) = \lambda_h^{(3)} \hat{u}_h^{(3)} + (\lambda_h^{(3)} u_h^{(3)} + h^2 f_2 + h^3 f_3)(1 - \chi) - h^{5/3} \chi' \frac{\partial}{\partial s} f_1 - h^{4/3} \chi'' f_1 - h^{-1/3} 2 \chi' \frac{\partial}{\partial s} z - h^{-2/3} \chi'' z = \lambda_h^{(3)} \hat{u}_h^{\text{appr}} + (h^2 f_2 + h^3 f_3)(1 - \chi) + h^{4/3} f_4 + \tilde{z}, \]
where function \( f_4 = h^{1/3} 2 \chi' \frac{\partial}{\partial s} f_1 - \chi'' f_1 \)

is bounded due (3.68) and has a small support:
\[ |f_4| \leq C, \]
\[ \sup(f_4(s, \eta)) = [h^{1/3}, 2h^{1/3}] \times [0, 1], \]

and function
\[ \tilde{z} = -h^{-1/3} 2 \chi' \frac{\partial}{\partial s} z - h^{-2/3} \chi'' z, \]
\[ \|\tilde{z}\|_{L^2(\Omega_h)} = O(h^m) \text{ for any } m \]
due to (3.68).

Now it is easy to estimate in \( L^2(\Omega_h) \) (in variables \( (x, y) \)) the discrepancy
\[-\Delta \hat{u}_{h}^{\text{appr}} - \lambda_{h}^{(3)} \hat{u}_{h}^{\text{appr}} \text{ via (3.84), (3.80), (3.82) and (3.84)}\]

\[
\| - \Delta \hat{u}_{h}^{\text{appr}} - \lambda_{h}^{(3)} \hat{u}_{h}^{\text{appr}} \|_{L^{2}(\Omega_{h})} = \\
= \|(h^{2} f_{2} + h^{3} f_{3})(1 - \chi) + h^{4/3} f_{4} + z\|_{L^{2}(\Omega_{h})} \leq \\
\leq h^{2}\|C\|_{L^{2}(\Omega_{h})} + h^{3}(h^{-2} + h^{-2\beta})\|C\|_{L^{2}(\Omega_{h} \cap \{s \in [1-h^{3}, 1]\})} + \\
+ h^{4/3}\|C\|_{L^{2}(\Omega_{h} \cap \{s \in [h^{1/3}, 2h^{1/3}]\})} \leq \\
\leq (h^{5/2} + h^{(3+\beta)/2} + h^{(7-3\beta)/2} + h^{2})C.
\]

Choosing \( \beta = 1 \) we obtain

\[
\| - \Delta \hat{u}_{h}^{\text{appr}} - \lambda_{h}^{(3)} \hat{u}_{h}^{\text{appr}} \|_{L^{2}(\Omega_{h})} \leq C h^{2}. \tag{3.86}
\]

It is well known that the set \( u_{i,h}, i \in \mathbb{N} \) of the orthonormalised eigenfunctions of \( A_{h} \) forms a basis in \( L^{2}(\Omega_{h}) \). Then \( \hat{u}_{h}^{\text{appr}} \) can be written in the form

\[
\hat{u}_{h}^{\text{appr}} = \sum_{i=1}^{\infty} c_{i} u_{i,h}. \tag{3.87}
\]

The main term of \( \hat{u}_{h}^{\text{appr}} \) is \( v_{0}(s) \sin h^{-1} \pi n, \ v_{0} \) is normalised, and other terms are of order \( O(h) \) or have relatively small support. Then one can easily check that

\[
\| \hat{u}_{h}^{\text{appr}} \|_{L^{2}(\Omega_{h})}^{2} = \sum_{i=1}^{\infty} c_{i}^{2} = \frac{1}{2} h + o(h). \tag{3.88}
\]

Substituting (3.87) into (3.80) we obtain

\[
\sum_{i=1}^{\infty} c_{i}^{2} (\lambda_{i,h} - \lambda_{h}^{(3)})^{2} \leq C h^{4}.
\]

Then from (3.88) follows that

\[
\min_{i} | \lambda_{i,h} - \lambda_{h}^{(3)} | \leq C h^{3/2}, \tag{3.89}
\]

which prove the validity of (4.44).

Let us denote by \( f_{h} \) the discrepancy \( -\Delta \hat{u}_{h}^{\text{appr}} - \lambda_{h}^{(3)} \hat{u}_{h}^{\text{appr}} \). Then \( f_{h} = \sum_{i=1}^{\infty} b_{i} u_{i,h} \), where \( \sum_{i=1}^{\infty} b_{i}^{2} \leq C h^{4} \). We can assume that \( \lambda_{h}^{(3)} \neq \lambda_{i,h} \) (the case \( \lambda_{h}^{(3)} = \lambda_{i,h} \) is trivial). Then \( c_{i} = (\lambda_{i,h} - \lambda_{h}^{(3)})^{-1} b_{i} \). Let us represent \( \hat{u}_{h}^{\text{appr}} \) as a

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The asymptotics \( \hat{u}_h^{\text{appr}} \) includes high order terms in its formula. They are required for the justification of (4.44), but not necessary for the approximation of eigenfunctions of \( A_h \). Consider the function \( u_h^{\text{appr}} = \chi(s h^{-1/3}) M g + (1 - \chi(s h^{-1/3}))(u_0 + h u_1) \). It is easy to see that it differs from \( \hat{u}_h^{\text{appr}} \) by a term of order \( h^{5/2} \) in \( L^2(\Omega_h) \) norm. Hence the second part of the statement of theorem follows.
Chapter 4

Models on graphs

In this chapter we obtain crucial results concerning the structure of the limiting problem on the graph. We show that in the non-critical case the more accurate model on the graph involves not Dirichlet, but ‘almost Dirichlet’ boundary conditions at the vertices, namely, the condition of the type

\[ V(0) + hTV'(0) = 0 \]  

at each vertex. Here \( T \) is a \( d \times d \) matrix, where \( d \) is the number of all edges incident to the vertex, \( V(0) \) is the vector \( (v_1(0), \ldots, v_d(0))^t \) of the values at the vertex that the function \( v \) attains along the edges, and \( V'(0) = (v'_1(0), \ldots, v'_d(0))^t \) is the vector of the values at the vertex of the derivatives taken along the edges taken in outgoing directions. This is important for the following reason. If one imposes Dirichlet boundary conditions at the vertices, the limiting problem on the graph splits into a number of disjoint problems on its edges, whereas boundary conditions (4.1) insure that the interaction between different edges exists although it is weak. The boundary conditions at the vertices for the limiting problem are fully defined by the scattering matrix at the threshold (the first transversal eigenvalue), which in turn is determined only by the geometry of the junction (in our simplified model by the angle of the slant). In the last section of the chapter we provide an explicit example for the case of zero-curvature, \( \kappa \equiv 0 \).
4.1 Limiting operator on graph

Let us consider the differential operator
\[
\hat{L}_h = L_0 + hL_1 = - \frac{d^2}{ds^2} - \frac{1}{4} \kappa'^2 + h \left( -\kappa \frac{d}{ds} - \frac{1}{4} \kappa'' \right)
\]
acting on the interval \((0, 1)\). It follows from \((3.16)\) and \((3.20)\) that
\[
\hat{L}_h(v_0 + hw_0) = \lambda_0 v_0 + h(\lambda_0 w_0 + \lambda_1 v_0 - \lambda_0 \kappa v_0) + h^2 L_1 w_0 = \]
\[
= (\lambda_0 + h\lambda_1)(1 - h\kappa)(v_0 + hw_0) +
+h^2 L_1 w_0 - h^2(\lambda_1 v_0 - \lambda_1 \kappa v_0 - \lambda_0 \kappa w_0) + h^3 \lambda_1 \kappa w_0.
\]
Notice that \(v_0, w_0 \in C^\infty([0, 1])\) (see the discussion in the beginning of the proof of Theorem \(3.5.1\)) and \(\kappa \in C^2([0, 1])\) by the assumptions of the present chapter. Notice also that
\[
e^{-h\kappa} = 1 - h\kappa + O(h^2),
\]
where the last term must be understood in terms of the norm \(L^\infty(0, 1)\). Then we obtain from \((4.2)\)
\[
\hat{L}_h(v_0 + hw_0) = (\lambda_0 + h\lambda_1)e^{-h\kappa}(v_0 + hw_0) + O(h^2),
\]
where the last term is understood in the norm \(L^\infty(0, 1)\). The operator \(\hat{L}_h\) cannot be symmetric, however we can slightly change it to obtain a symmetric operator. Indeed,
\[
\frac{d}{ds} \left( e^{h\kappa} \frac{d}{ds} \right) = e^{h\kappa} \left( \frac{d^2}{ds^2} + h\kappa' \frac{d}{ds} \right).
\]
Then multiplying \((4.3)\) by \(e^{h\kappa}\) we obtain
\[
L_h(v_0 + hw_0) = (\lambda_0 + h\lambda_1)(v_0 + hw_0) + O(h^2),
\]
where
\[
L_h = e^{h\kappa} \hat{L}_h = - \frac{d}{ds} \left( e^{h\kappa} \frac{d}{ds} \right) - \frac{1}{4} e^{h\kappa}(\kappa'^2 + h\kappa'').
\]
In view of \((4.4)\) it is natural to try to approximate the function
\[
\bar{v}_h = v_0 + hw_0
\]
by some eigenfunction of the operator \(L_h\) with appropriate boundary conditions,
which we will derive first. From (3.52), (3.53) we have
\[ \tilde{v}_h(0) = C_1 + C_2 + h(C_3 + C_4), \]
\[ \tilde{v}_h'(0) = (C_1 - C_2) i \sqrt{\lambda_0} + h \left( (C_1 - C_2) \frac{i\lambda_1}{2\sqrt{\lambda_0}} + (C_3 - C_4) i \sqrt{\lambda_0} \right). \] (4.6)

Then for the non-critical case, \( \lambda_0 < 0 \), via (3.57) and (3.60) we obtain
\[ \tilde{v}_h(0) = -h^2 i \sqrt{\lambda_0} \sigma C_1, \]
\[ \tilde{v}_h'(0) = 2 i \sqrt{\lambda_0} C_1 + h \left( \frac{i\lambda_1}{\sqrt{\lambda_0}} - 2 i \lambda_0 \sigma \right) C_1. \]

Hence
\[ \tilde{v}_h(0) + h \sigma \tilde{v}_h'(0) = h^2 \left( \frac{i\lambda_1}{\sqrt{\lambda_0}} - 2 i \lambda_0 \sigma \right) \sigma C_1. \] (On the right end we obviously have \( \tilde{v}_h(1) = 0 \).) This suggests that we need to consider the self-adjoint operator \( L_h \) acting in \( L^2(0,1) \), given by (4.42) with boundary conditions
\[ v(0) + h \sigma v'(0) = 0, \]
\[ v(1) = 0. \] (4.7)

Ideally, we would like to describe the asymptotic solution to spectral problem (3.2) in terms of the eigenvalues and eigenfunctions of \( L_h \).

We slightly modify the function \( \tilde{v}_h \) so that it would satisfy boundary conditions (4.7). Consider the function
\[ \hat{v}_h = \tilde{v}_h + h^2 N(e^{i\sqrt{\lambda_0}s} + e^{-i\sqrt{\lambda_0}s}) \chi(4s), \] (4.8)
where
\[ N = -\left( \frac{i\lambda_1}{2\sqrt{\lambda_0}} - i \lambda_0 \sigma \right) \sigma C_1, \]
and \( \chi \) is from (3.76). Clearly, \( \hat{v}_h \) satisfies (4.7). Moreover, \( \hat{v}_h \) is approximate solution to the eigenvalue problem,
\[ L_h \hat{v}_h = (\lambda_0 + h \lambda_1) \hat{v}_h + O(h^2). \] (4.9)

The last term is understood in the norm \( L^\infty(0,1) \) and, hence, it is \( O(h^2) \) in
$L^2(0, 1)$. Consider the spectral problem for the operator $L_h,$

\[
L_h v^{(k)}_{h} = \mu^{(k)}_{h} v^{(k)}_{h},
\]

\[
v^{(k)}_{h}(0) + h\sigma \frac{d}{dx}v^{(k)}_{h}(0) = 0, \quad v^{(k)}_{h}(1) = 0.
\]

Applying absolutely the same reasoning as in the previous section when estimates (3.89), (3.90) have been derived, we conclude that

\[
\min_{k} |\mu^{(k)}_{h} - (\lambda_0 + h\lambda_1)| \leq Ch^2,
\]

\[
\| \sum_{k \in K_d} c_k(h)v^{(k)}_{h} - \hat{v}_h \|_{L^2(0,1)} \leq Cd^{-1}h^2,
\]

(4.10)

where $\mu^{(k)}_{h}$ and $v^{(k)}_{h}$ are the eigenvalues and corresponding eigenfunctions of $L_h$, $C$ is an $h$-independent constant, and summation is taken over the set of indices $K_d$ such that $|\mu^{(k)}_{h} - (\lambda_0 + h\lambda_1)| \leq d$.

The case of the positive $\lambda_0$ is analogous, and one can easily obtain (4.10) for function (4.8) with $N$ given by

\[
N = - \left( \frac{i\lambda_1}{2\sqrt{-\lambda_0}} + \lambda_0 \sigma \right) \sigma C_1.
\]

Estimates (4.10) show that all the asymptotics $\lambda_0 + h\lambda_1$ and $v_0 + hw_0$ can be approximated by the eigenvalues and eigenfunctions of $L_h$. Moreover, an almost converse statement is valid. But in order to prove this we need to obtain more precise information about the spectrum of $L_h$. This is the main purpose of the next section.

### 4.2 Spectrum of the limiting operator $L_h$

We are interested in the behaviour of the spectrum of $L_h$ as $h \to 0$, in particular, in its relation to the spectrum of the operator $L_0$,

\[
L_0 v = -v'' - \frac{1}{4}\kappa^2 v,
\]

\[
D(L_0) = H^1_0 \cap H^2 = H^1_0(0, 1) \cap H^2(0, 1).
\]

**Remark 4.2.1.** Notice that if $\sigma = 0$ (i.e. $s = -1$), boundary conditions (4.7) are purely Dirichlet. In this case the operator $L_h$ is a regular perturbation of
the operator $L_0$. Then the spectrum of $L_h$ converges to the spectrum of $L_0$. So we assume in the following that $\sigma \neq 0$.

We consider first the operator $L_{h,0}$ defined by the same differential operation as $L_0$, 

$$L_{h,0} v = -v'' - \frac{1}{4} \kappa^2 v,$$

with the domain $D(L_{h,0})$ that consists of all $v \in H^2$ satisfying boundary conditions (4.7).

The operators $L_0$, $L_{h,0}$, $L_h$ are self-adjoint and their spectra are discrete with the only limiting point at infinity. Integration by parts yields that the following bilinear forms correspond to the operators:

$$\Lambda_0(v, w) = \int_0^1 v'w'ds - \int_0^1 \frac{1}{4} \kappa^2 vwds, \, v, w \in H^1_0 = H^1_0(0,1),$$

$$\Lambda_{h,0}(v, w) = \int_0^1 v'w'ds - \int_0^1 \frac{1}{4} \kappa^2 vwds - \frac{1}{h\sigma} v(0)w(0), \, v, w \in H^1_0,$$

$$\Lambda_h(v, w) = \int_0^1 e^{h\kappa}v'w'ds - \int_0^1 \frac{1}{4} e^{h\kappa}(\kappa^2 + h\kappa'')vwdx - \frac{1}{h\sigma} v(0)w(0), \, v, w \in H^1_0(0)$$

respectively, where $H^1_{(0)}$ is the set of functions from $H^1$ vanishing at 1, $v(1) = 0$.

We use the minimax definition for eigenvalues of an operator $L$ with a bilinear form $\Lambda$:

$$\mu^{(k)} := \inf_{\dim W = k} \sup_{v \in W} \frac{\Lambda(v, v)}{\|v\|^2}, \, k = 1, 2, \ldots,$$  \hspace{1cm} (4.12)

where $W$ denotes a subspace of the domain of the bilinear form, and $\|v\|^2$ denotes $\int_0^1 v'^2ds$ for short.

**Lemma 4.2.2.** The eigenvalues of $L_{h,0}$ and $L_0$ alternate:

$$\mu_{h,0}^{(k)} < \mu_0^{(k)} < \mu_{h,0}^{(k+1)}, \, k = 1, 2, \ldots.$$  

If $\sigma < 0$, then

$$\lim_{h \to 0} \mu_{h,0}^{(k)} = \mu_0^{(k)}, \, k = 1, 2, \ldots.$$
If $\sigma > 0$, then

$$\mu_{h,0}^{(1)} = -\frac{1}{(h\sigma)^2} + O(h^m), \forall m > 0,$$

$$\lim_{h \to 0} \mu_{h,0}^{(k+1)} = \mu_0^{(k)}, \ k = 1, 2, \ldots,$$

and the first eigenfunction of $L_{h,0}$ can be approximated by the exponentially decaying function $f_h$:

$$\|v_{h,0}^{(1)} - f_h\| = O(h^m), \forall m > 0,$$

where $f_h$ is given by

$$f_h = \frac{\exp((-h\sigma)^{-1}s)\chi(2s/s_0)}{\|\exp((-h\sigma)^{-1}s)\chi(2s/s_0)\|}.$$  

(4.15)

Proof. Since for an arbitrary function $v \in H^1_0$ we have $\Lambda_0(v,v) = \Lambda_{h,0}(v,v)$, it follows from the minimax principle that

$$\mu_{h,0}^{(k)} \leq \mu_0^{(k)}.$$

Let us consider the spectrum of the self-adjoint operator $L^\gamma$ corresponding to the bilinear form

$$\Lambda^\gamma(v, w) = \int_0^1 v'w'ds - \int_0^1 \frac{1}{4} \kappa^2 vwds + \gamma v(0)w(0)$$

defined on $H^1_{(0)}$, where the parameter $\gamma \in \mathbb{R}^n$. Denote its eigenvalues by $\mu^{(k)}(\gamma)$. We have $\mu^{(k)}(\gamma) = \mu_{h,0}^{(k)}$ (and the equality of the corresponding eigenfunctions) provided that $\gamma = -\frac{1}{h\sigma}$. Since for any fixed $v$ the bilinear form $\Lambda^\gamma(v, v)$ is a continuous non-decreasing function of $\gamma$, each eigenvalue $\mu^{(k)}(\gamma)$ is a continuous non-decreasing function of $\gamma$ as well. Let us fix some $\mu \in \mathbb{R}$. It follows from the theory of ordinary differential equations that if $v \in H^1_{(0)}, v \neq 0$, is some solution of the equation

$$-v'' - \frac{1}{4} \kappa^2 v = \mu v,$$

then any other solution of (4.16) from $H^1_{(0)}$ is given by $Cv$, where $C$ is constant. For any $\mu \in \mathbb{R}$ there exists a solution (4.16) from $H^1_{(0)}$, therefore $\mu$ is an eigenvalue of either $L^\gamma$ (for some particular value of $\gamma$) or $L_0$. These observations imply several important consequences. Firstly, each eigenvalue $\mu^{(k)}(\gamma)$ is a
continuous strictly increasing function of $\gamma$; secondly,

$$\mu^{(k)}(\gamma_1) < \mu^{(k)}_0 < \mu^{(k+1)}(\gamma_2), \forall \gamma_1, \gamma_2 \in \mathbb{R}, \ k = 1, 2, \ldots; \quad (4.17)$$

and, thirdly,

$$\lim_{\gamma \to +\infty} \mu^{(k)}(\gamma) = \mu^{(k)}_0 = \lim_{\gamma \to -\infty} \mu^{(k+1)}(\gamma), \ k = 1, 2, \ldots.$$ 

The statements of the lemma follow immediately except the one concerning $\mu^{(1)}_{h,0}$ and $v_{h,0}^{(1)}$ when $\sigma > 0$.

The function $f_h$ belongs to the domain of $L_{h,0}$ and satisfies the following equation:

$$L_{h,0}f_h = -f''_h = -(h\sigma)^{-2}f_h + O((h\sigma)^{-3/2}\exp(-(h\sigma)^{-1}s_0/2)). \quad (4.18)$$

Then it is easy to show in the way absolutely analogous to the proof of error bounds (3.89), (3.90) the validity of the first equality in (4.13) and asymptotics (4.14).

**Remark 4.2.3.** Notice that if $\sigma > 0$, then equally to the case of the operator $L_{h,0}$ we have the following asymptotics for the first eigenvalue and the corresponding eigenfunction of $L_h$:

$$L_h f_h = -f''_h = -(h\sigma)^{-2}f_h + O(h^m),$$

$$\mu_h^{(1)} = -\frac{1}{(h\sigma)^2} + O(h^m),$$

$$\|v_h^{(1)} - f_h\| = O(h^m), \forall m > 0.$$ \hspace{1cm} (4.19)

The next lemma establishes asymptotic proximity of the eigenvalues of $L_h$ and $L_{h,0}$. In turn this will provide the desired result on the convergence of the eigenvalues of $L_h$ to the eigenvalues of $L_0$.

**Lemma 4.2.4.** The eigenvalues of the operators $L_h$ and $L_{h,0}$ are asymptotically close:

$$\lim_{h \to 0} |\mu^{(k)}_{h,0} - \mu^{(k)}_h| = 0, \ k = 1, 2, \ldots. \quad (4.20)$$
Proof. Let us consider the difference between the bilinear forms $\Lambda_h$ and $\Lambda_{h,0}$:

$$|\Lambda_h(v, v) - \Lambda_{h,0}(v, v)| = \left| \int_0^1 (e^{h\kappa} - 1)(v')^2 ds - \int_0^1 \frac{1}{4} (e^{h\kappa}(\kappa^2 + h\kappa'') - \kappa^2)v^2 ds \right|.$$ 

It follows immediately from (4.11) that if $\sigma < 0$ then

$$|\Lambda_h(v, v) - \Lambda_{h,0}(v, v)| \leq h C(\Lambda_{h,0}(v, v) + C\|v\|^2), \quad (4.21)$$

where $C$ is some constant independent of $h$. These estimates allow us to derive the statement of the lemma using the minimax definition of the eigenvalues. Indeed, from (4.21) we obtain

$$\Lambda_h(v, v) \leq \Lambda_{h,0}(v, v) + |\Lambda_h(v, v) - \Lambda_{h,0}(v, v)| \leq \Lambda_{h,0}(v, v) + h C(\Lambda_{h,0}(v, v) + C\|v\|^2),$$

$$\Lambda_{h,0}(v, v) \leq \Lambda_h(v, v) + |\Lambda_h(v, v) - \Lambda_{h,0}(v, v)| \leq \Lambda_h(v, v) + h C(\Lambda_{h,0}(v, v) + C\|v\|^2).$$

Then it follows from definition (4.12) that

$$\mu_{h,0}^{(k)} - h C(\mu_{h,0}^{(k)} + C) \leq \mu_h^{(k)} \leq \mu_{h,0}^{(k)} + h C(\mu_{h,0}^{(k)} + C).$$

The latter implies (4.20) for the case $\sigma < 0$.

If $\sigma > 0$ the matter is more complicated. In this case we need to employ the following lemma, whose proof we provide later.

**Lemma 4.2.5.** Let $v$ be the following linear combination of the eigenfunction of either $L_h$:

$$v = \sum_{i=2}^k a_i v_h^{(i)},$$

or $L_{h,0}$:

$$v = \sum_{i=2}^k a_i v_{h,0}^{(i)}.$$

Then for such $v$ inequality (4.21) is valid.

Notice that $f_h$ is almost orthogonal to the eigenfunctions of $L_h$ and $L_{h,0}$.
4.19 4.26 4.18

assumed in the above that $v$ is as in Lemma 4.25. Notice also the following obvious observation:

\[ \Lambda_h(f_h, v) = (L_h f_h, v) = O(h^m) \|v\|, \; \forall m > 0, \]

\[ \Lambda_{h,0}(f_h, v) = (L_{h,0} f_h, v) = O(h^m) \|v\|, \; \forall m > 0, \]

where $v$ is as in Lemma 4.25. Notice also the following obvious observation:

\[ \Lambda_h(f_h, f_h) = -\frac{1}{h^2 \sigma^2} + O(h^m) = \mu_h^{(1)} + O(h^m), \]

\[ \Lambda_{h,0}(f_h, f_h) = -\frac{1}{h^2 \sigma^2} + O(h^m) = \mu_{h,0}^{(1)} + O(h^m), \; \forall m > 0. \]

Then from (4.23) and (4.24) we derive

\[ \Lambda_h(a_1 f_h + v, a_1 f_h + v) = a_1^2 \Lambda_h(f_h, f_h) + 2a_1 \Lambda_h(f_h, v) + \Lambda_h(v, v) \leq \]

\[ \leq \Lambda_h(v, v) + O(h^m) \|a_1 f_h + v\|^2, \; \forall m > 0, \]

and analogously

\[ \Lambda_{h,0}(a_1 f_h + v, a_1 f_h + v) \leq \Lambda_{h,0}(v, v) + O(h^m) \|a_1 f_h + v\|^2, \; \forall m > 0. \]

Let $W_k$ be a subspace of $H^1_{(0)}$ spanned by the functions $f_h, v^{(2)}_{h,0}, \ldots, v^{(k)}_{h,0}$ and let $W_{k-1}$ be a subspace of $H^1_{(0)}$ spanned by the functions $v^{(2)}_{h,0}, \ldots, v^{(k)}_{h,0}$. Clearly $\dim W_k = k$ since $f_h$ is ‘very close’ to $v^{(1)}_{h,0}$, see (4.14). Let $v \in W_k$. Then the function $v$ can be presented in the form $v = a_1 f_h + \tilde{v}$, where $\tilde{v} \in W_{k-1}$, and $\|v\|^2 = \|a_1 f_h\|^2 + \|\tilde{v}\|^2 + O(h^m) \|a_1 f_h\| \|\tilde{v}\|, \; \forall m > 0$, due to (4.22). Hence we obtain from (4.25) that

\[ \frac{\Lambda_h(v, v)}{\|v\|^2} \leq \frac{\Lambda_h(\tilde{v}, \tilde{v})}{\|\tilde{v}\|^2} + O(h^m) \leq \frac{\Lambda_h(\tilde{v}, \tilde{v})}{\|\tilde{v}\|^2} + O(h^m). \]

That is for any $v \in W_k$ there exists $\tilde{v} \in W_{k-1}$ such that (4.26) holds. (We assumed in the above that $v \neq a_1 f_h$, otherwise the reasoning is trivial.) Then
via (4.12), (4.21) and (4.26) we derive for \( k \geq 2 \) that

\[
\mu_h^{(k)} \leq \sup_{v \in W_k} \frac{\Lambda_h(v, v)}{\|v\|^2} \leq \sup_{v \in W_{k-1}} \frac{\Lambda_h(v, v)}{\|v\|^2} + O(h^m) \leq \sup_{v \in W_{k-1}} \frac{\Lambda_{h,0}(v, v) + hC(\Lambda_{h,0}(v, v) + C\|v\|^2)}{\|v\|^2} + O(h^m) = \mu_h^{(k)}(1 + hC) + O(h).
\]

(4.27)

Analogously, let \( W_k \) be a subspace of \( H^1_{(0)} \) spanned by the functions \( f_h, v_h^{(2)}, \ldots, v_h^{(k)} \) and \( W_{k-1} \) be a subspace of \( H^1_{(0)} \) spanned by the functions \( v_h^{(2)}, \ldots, v_h^{(k)} \). Then for \( k \geq 2 \)

\[
\mu_{h,0}^{(k)} \leq \sup_{v \in W_k} \frac{\Lambda_h(v, v)}{\|v\|^2} \leq \sup_{v \in W_{k-1}} \frac{\Lambda_{h,0}(v, v) + hC(\Lambda_{h,0}(v, v) + C\|v\|^2)}{\|v\|^2} + O(h^m) = \mu_{h,0}^{(k)} + hC\mu_h^{(k)} + O(h).
\]

(4.28)

Estimates (4.27) and (4.28) imply the statement of the lemma. \( \square \)

**Proof of Lemma 4.2.5.** In order to proof the validity of (4.21) we need to estimate somehow the term \(-\frac{1}{h^3}v^2(0)\) for the eigenfunctions \( v_h^{(k)} \) and \( v_{h,0}^{(k)} \), \( k = 2, 3, \ldots \).

The following reasoning is equally valid for both \( v_h^{(k)} \) and \( v_{h,0}^{(k)} \), so we give it only for the eigenfunctions of \( L_h \). Since \( \kappa(s) = 0 \) for \( s \in [0, s_0] \),

\[
-\frac{d^2}{ds^2}v_{h}^{(k)} = \mu_{h}^{(k)}v_{h}^{(k)}, \quad s \in [0, s_0].
\]

(4.29)

Let \( \mu_{h}^{(k)} \) be positive. Then

\[
v_{h}^{(k)} = a_1 \sin \left( \sqrt{\mu_{h}^{(k)}} \right) + a_2 \cos \left( \sqrt{\mu_{h}^{(k)}} \right), \quad s \in [0, s_0].
\]

(4.30)

From boundary conditions (4.7) it follows that

\[
a_2 = -h \sigma \sqrt{\mu_{h}^{(k)}} a_1.
\]

(4.31)
The coefficient $a_1$ is determined by the norm $\|v_h^{(k)}\|$. Indeed,

$$\|v_h^{(k)}\|^2 \geq a_1^2 \int_0^\infty \left[ \sin^2 \left( \sqrt{\mu_h^{(k)}} s \right) + h^2 \sigma^2 \mu_h^{(k)} \cos^2 \left( \sqrt{\mu_h^{(k)}} s \right) - 2h \sigma \sqrt{\mu_h^{(k)}} \sin \left( \sqrt{\mu_h^{(k)}} s \right) \cos \left( \sqrt{\mu_h^{(k)}} s \right) \right] ds =$$

$$= a_1^2 \left[ \frac{s_0}{2} \left(1 + h^2 \sigma^2 \mu_h^{(k)}\right) + \frac{1}{4 \sqrt{\mu_h^{(k)}}} \sin \left(2 \sqrt{\mu_h^{(k)}} s_0 \right) + \frac{h \sigma}{2} \left( \cos \left(2 \sqrt{\mu_h^{(k)}} s_0 \right) - 1 \right) \right].$$

(4.32)

If the eigenvalue $\mu_h^{(k)}$ is bounded away from zero uniformly with respect to $h$, $\mu_h^{(k)} > C > 0$, we conclude from (4.32) that

$$\|v_h^{(k)}\|^2 \geq a_1^2 C,$$

for some $C > 0$, and hence

$$|a_1| \leq C \|v_h^{(k)}\|,$$

(4.33)

where $C$ independent of $h$. If for some sequence $h \rightarrow 0$ we have $|\mu_h^{(k)}| \rightarrow 0$, then by applying Taylor expansion to the right hand side of (4.32) we arrive at

$$\|v_h^{(k)}\|^2 \geq a_1^2 \left[ \frac{1}{3} \mu_h^{(k)} s_0^3 + O \left( h \mu_h^{(k)} + \left( \mu_h^{(k)} \right)^2 \right) \right].$$

Therefore,

$$|a_1| \leq \frac{C \|v_h^{(k)}\|}{\mu_h^{(k)}}$$

(4.34)

for small enough $h$ and some constant $C$ independent of $h$.

At this stage we need to use the following proposition whose proof we provide after the proof of the lemma.

**Proposition 4.2.6.** Each eigenvalue $\mu_h^{(k)}$, $k = 2, 3, \ldots$ of $L_h$ is bounded uniformly with respect to $h$, i.e. for any $k = 2, 3, \ldots$ there exists a constant $C$ such that

$$|\mu_h^{(k)}| \leq C.$$

(4.35)

We obtain from (4.30), (4.31) and either (4.34) or (4.33) and Proposition 4.2.6.
that
\[ |v_h^{(k)}(0)| \leq Ch\|v_h^{(k)}\|, \quad k = 2, 3, \ldots \tag{4.36} \]

The case of negative \( \mu_h^{(k)} \) is very similar. In this instance
\[ v_h^{(k)} = a_1 \exp \left( \sqrt{|\mu^{(k)}_h|} s \right) + a_2 \exp \left( -\sqrt{|\mu^{(k)}_h|} s \right), \quad s \in [0, s_0]. \tag{4.37} \]

From boundary conditions (4.7) it follows that
\[ a_1 + a_2 = -h \sigma \sqrt{|\mu_h^{(k)}|} (a_1 - a_2). \tag{4.38} \]

Denote \( b_1 = (a_1 - a_2)/2 \) and \( b_2 = (a_1 + a_2)/2 \). Then
\[
\|v_h^{(k)}\|^2 \geq b_1^2 \int_0^{s_0} \left[ \left( 1 - h \sqrt{|\mu_h^{(k)}|} \sigma \right)^2 \exp \left( 2 \sqrt{|\mu_h^{(k)}|} s \right) + \right. \\
+ \left( 1 + h \sqrt{|\mu_h^{(k)}|} \sigma \right)^2 \exp \left( -2 \sqrt{|\mu_h^{(k)}|} s \right) - 2 \left( 1 - h^2 \mu_h^{(k)} \sigma^2 \right) \right] ds = \\
b_1^2 \left[ \frac{1}{2 \sqrt{|\mu_h^{(k)}|}} \left( 1 - h \sqrt{|\mu_h^{(k)}|} \sigma \right)^2 \exp \left( 2 \sqrt{|\mu_h^{(k)}|} s \right) \right]_{0}^{s_0} + \\
- \frac{1}{2 \sqrt{|\mu_h^{(k)}|}} \left( 1 + h \sqrt{|\mu_h^{(k)}|} \sigma \right)^2 \exp \left( -2 \sqrt{|\mu_h^{(k)}|} s \right) \right]_{0}^{s_0} - \\
- 2 \left( 1 - h^2 \mu_h^{(k)} \sigma^2 \right) s_0. \tag{4.39} \]

If the eigenvalue \( \mu_h^{(k)} \) is bounded away from zero uniformly with respect to \( h \), \( |\mu_h^{(k)}| > C > 0 \), we arrive at
\[ |b_1| \leq C \|v_h^{(k)}\|, \]
where \( C \) independent of \( h \). Then (4.36) follows from (4.37), (4.38) and Proposition [4.2.6].

If for some sequence \( h \to 0 \) we have \( |\mu_h^{(k)}| \to 0 \), then by applying Taylor expansion to the right hand side of (4.39) we arrive at
\[ \|v_h^{(k)}\|^2 \geq b_1^2 \left[ \frac{4}{3} |\mu_h^{(k)}| s_0^3 + O \left( h^{\mu_h^{(k)}} + (\mu_h^{(k)})^2 \right) \right]. \]
Therefore,

\[ |b_1| \leq \frac{C}{\sqrt{|\mu_h^{(k)}|}} \|v_h^{(k)}\|, \]

for small enough $h$ and some constant $C$ independent of $h$, and (4.36) follows from (4.37), (4.38).

The case $\mu_h^{(k)} = 0$ is trivial. For $s \in [0, s_0]$ we have

\[ v_h^{(k)} = a_1 s - h\sigma a_1. \]

Then

\[ \|v_h^{(k)}\|^2 \geq \int_0^{s_0} (a_1 s - h\sigma a_1)^2 ds \geq C a_1^2, \]

and (4.36) follows.

Analogously for the eigenfunctions of $L_{h,0}$ we have

\[ |v_h^{(k)}(0)| \leq C h \|v_h^{(k)}\|, \quad k = 2, 3, \ldots \] (4.40)

**Remark 4.2.7.** Constants in (4.36), (4.40) are uniform in $h$, however may depend on $k$.

The crucial estimates (4.36), (4.40) are valid not only for the eigenfunctions of $L_h$ and $L_{h,0}$ but also for their finite linear combinations. Indeed, let

\[ v = \sum_{i=2}^{k} a_i v_h^{(i)}. \] (4.41)

Then

\[ v^2(0) \leq C \sum_{i=2}^{k} a_i^2 (v_h^{(i)}(0))^2 \leq C h^2 \sum_{i=2}^{k} a_i^2 \|v_h^{(i)}\|^2 = C h^2 \|v\|^2, \]

and analogously for $v = \sum_{i=2}^{k} a_i v_{h,0}^{(i)}$. From the latter estimate it follows, that in the case of positive $\sigma$ inequalities (4.21) are valid for such linear combinations of the eigenfunctions. \qed

**Proof of Proposition 4.2.6.** Let us introduce two self-adjoint operators $\mathcal{L}_h$, $\mathcal{L}_{h,\gamma}$.
via their bilinear forms:

\[
\Psi_h(v, w) := \int_0^1 e^{h\kappa} v' w' ds - \int_0^1 \frac{1}{4} e^{h\kappa}(\kappa^2 + h\kappa'')vw ds,
\]

\[
\Psi_{h,\gamma}(v, w) := \int_0^1 e^{h\kappa} v' w' ds - \int_0^1 \frac{1}{4} e^{h\kappa}(\kappa^2 + h\kappa'')vw ds + \gamma v(0)w(0),
\]

with the domains

\[
D(\Psi_h) = H^1_0,
D(\Psi_{h,\gamma}) = H^1_{(0)}.
\]

The corresponding eigenvalue problems for these operators are given by the equation

\[
-\frac{d}{ds} \left( e^{h\kappa} \frac{d}{ds} v \right) - \frac{1}{4} e^{h\kappa}(\kappa^2 + h\kappa'')v = \omega_h v, \ s \in (0, 1),
\]

accompanied by Dirichlet boundary conditions for \( L_h \) or the conditions

\[
\gamma v(0) - v'(0) = 0,
\]

\[
v(1) = 0,
\]

for \( L_{h,\gamma} \). Analogously to the case of the operators \( L_\gamma \) and \( L_0 \) one can show that for every fixed \( h \)

\[
\omega^{(k)}_h(\gamma) < \omega^{(k)}_h < \omega^{(k+1)}_h(\gamma), \ \gamma \in \mathbb{R}, \ k = 1, 2, \ldots,
\]

where \( \omega^{(k)}_h(\gamma) \) and \( \omega^{(k)}_h(\gamma) \) are the eigenvalues of \( L_h \) and \( L_{h,\gamma} \) arranged in non-decreasing order. In particular, when \( \gamma = -\frac{1}{hs\sigma} \) (then \( \omega^{(k)}_h(\gamma) = \mu^{(k)}_h \)), we have

\[
\mu^{(k)}_h < \omega^{(k)}_h < \mu^{(k+1)}_h, \ k = 1, 2, \ldots
\]

The operator \( L_h \) is a regular perturbation of \( L_0 \), hence \( \omega^{(k)}_h \to \mu^{(k)}_0 \). Therefore for each \( k \geq 2 \) the eigenvalue \( \mu^{(k)}_h \) is bounded uniformly with respect to \( h \).

Combining Lemmas 4.2.2 and 4.2.4 and Remark 4.2.1 we obtain the description of the asymptotic behaviour of the spectrum of \( L_h \).

**Theorem 4.2.8.** Let \( \sigma \leq 0 \), then

\[
\lim_{h \to 0^+} \mu^{(k)}_h = \mu^{(k)}_0, \ k = 1, 2, \ldots.
\]
Let \( \sigma > 0 \), then
\[
\mu_h^{(1)} = -\frac{1}{(h\sigma)^2} + O(h^m), \forall m > 0,
\]
\[
\lim_{h \to 0} \mu_h^{(k+1)} = \mu_0^{(k)}, k = 1, 2, \ldots.
\]

### 4.3 Approximation of the problem in \( \Omega_h \) by the limiting problem on graph

Theorem 4.2.8 together with \((4.10)\) imply the following

**Theorem 4.3.1.** Let \( \mu_h^{(k)} \) be the \( k \)-th eigenvalue of the operator \( L_h \) such that \( \lim_{h \to 0} \mu_h^{(k)} \neq 0 \), \( k = 1, 2, \ldots \) if \( \sigma < 0 \), \( k = 2, 3, \ldots \) if \( \sigma > 0 \). Then for \( \lambda_h^{(k)} \) - \( k \)-th eigenvalue of problem \((3.16)\), \((3.58)\), and \( \lambda_1^{(k)} \), given by \((3.64)\) with \( \lambda_0 = \lambda_0^{(k)} \), we have
\[
|\mu_h^{(k)} - (\lambda_0^{(k)} + h\lambda_1^{(k)})| \leq Ch^2, \quad k = 1, 2, \ldots \quad \text{if} \quad \sigma < 0,
\]
\[
|\mu_h^{(k)} - (\lambda_0^{(k-1)} + h\lambda_1^{(k-1)})| \leq Ch^2, \quad k = 2, 3, \ldots \quad \text{if} \quad \sigma > 0.
\]

Let \( v_h^{(k)} \) be the eigenfunction corresponding to \( \lambda_0^{(k)} \), and \( w_h^{(k)} \) be the solution of \((3.20)\) satisfying the conditions described in Section \(3.4\). Then
\[
\|v_h^{(k)} - (v_0^{(k)} + hw_0^{(k)})\|_{L^2(0,1)} \leq Ch^2, \quad k = 1, 2, \ldots \quad \text{if} \quad \sigma < 0,
\]
\[
\|v_h^{(k)} - (v_0^{(k-1)} + hw_0^{(k-1)})\|_{L^2(0,1)} \leq Ch^2, \quad k = 2, 3, \ldots \quad \text{if} \quad \sigma > 0.
\]

**Proof.** The first part of the statement of the theorem follows immediately from Theorem 4.2.8 and the first inequality in \((4.10)\) (notice that \( \lambda_0^{(k)} = \mu_0^{(k)} \)). Theorem 4.2.8 implies also that for any \( \lambda_0^{(k)} \) there exists constant \( d \) such that for small enough \( h \) the \( d \)-neighbourhood of \( \lambda_0^{(k)} + h\lambda_1^{(k)} \) contains exactly one eigenvalue \( \mu_h^{(k)} \). Then the second part of the statement follows from \((4.10)\) and a simple observation that the function \( \tilde{v}_h^{(k)} = v_0^{(k)} + hw_0^{(k)} \) differs from \( \tilde{v}_h^{(k)} \) only by the term of order \( h^2 \).

**Remark 4.3.2.** The limiting operator \( L_h \) corresponds to the non-critical case. The critical case is different from the non-critical one. The reason is that in the critical case the limiting operator on the graph must include a spectral parameter in the boundary condition. Indeed, for example if \( \lambda_0 \) is negative, via \((4.6)\) and \((3.70)\) we have
\[
-h\lambda_0\sigma\tilde{v}_h(0) + \tilde{v}_h'(0) = -h^22\lambda_0^{3/2}\sigma^2C_1.
\]
The study of this problem goes beyond the scope of the present thesis.

Now we can prove that the limiting problem on the graph gives the correct asymptotics of the problem in the thin domain $\Omega$. Indeed, it is easy to see via (4.43) that
\[ \| (u_0 + hu_1) - (\varphi_0 + h\kappa\varphi_1)v_h \|_{L^2(\Omega_h)} \leq Ch^{5/2}. \]
Then the assertion follows from Theorem 3.5.1 and Theorem 4.3.1 immediately:

**Theorem 4.3.3.** Let $\mu^{(k)}_h$ be as in Theorem 4.3.1 and $v^{(k)}_h$ be the corresponding normalised eigenfunction. Then there exists $h_0 > 0$ and a constant $C$ independent of $h$ such that for any $0 < h \leq h_0$ there exist an eigenvalue $\lambda_{i,h}$ of the operator $A_h$ such that
\[ |\lambda_{i,h} - (h^{-2}\lambda_{-2} + \mu^{(k)}_h)| \leq Ch^{3/2}. \] (4.44)

Moreover, the function $(\varphi_0 + h\kappa\varphi_1)v^{(k)}_h$ approximates eigenfunctions $u_{i,h}$ of the operator $A_h$ outside a small neighbourhood of the origin. Namely, let $\Theta_h$ be a characteristic function of the set $(0, 2h^{1/3})^2$, then for any $d > 0$ and any $0 < h \leq h_0$ there exist coefficients $c_i(h)$ such that
\[ \left\| \left( u_{i,h}^{\text{appr}} - \sum_{|\lambda_{i,h} - (h^{-2}\lambda_{-2} + \mu^{(k)}_h)| \leq d} c_i(h)u_{i,h} \right) (1 - \Theta_h) \right\|_{L^2(\Omega_h)} \leq Cd^{-1}h^2. \]

**Remark 4.3.4.** In the last two chapters we never essentially used the assumption about the boundary conditions imposed on the slanted end $\gamma_1$ of the domain $\Omega_h$. All the arguments and results obtained are equally valid for the eigenvalue problem
\[ -\Delta u_h = \lambda_h u_h, \quad x \in \Omega_h, \]
\[ u = 0, \quad x \in \partial \Omega_h, \] (4.45)
except that in this setting the critical case is not possible. Let $u^N_h$ be the eigenfunction of problem (3.2) and $u^D_h$ be the eigenfunction of problem (4.45). Then a symmetric extension of $u^N_h$ into $\hat{\Omega}_h$ (see Introduction and Figure 0-4 in particular), and antisymmetric extension of $u^D_h$ into $\hat{\Omega}_h$ are eigenfunctions of the problem
\[ -\Delta u_h = \lambda_h u_h, \quad x \in \hat{\Omega}_h, \]
\[ u = 0, \quad x \in \partial \hat{\Omega}_h. \] (4.46)
Denote by $v^N_h(s)$ and $v^D_h(s)$ the eigenfunction of the limiting operators for problems (3.2) and (4.45) respectively. They satisfy the following boundary conditions at the vertex:

\[
\begin{align*}
v^N_h(0) + h \sigma^N \frac{d}{ds} v^N(0) &= 0, \\
v^D_h(0) + h \sigma^D \frac{d}{ds} v^D(0) &= 0.
\end{align*}
\]

(In particular $\sigma^N = \sigma$ as is in (3.55), and $\sigma^D$ is given by the same formula with $\hat{s}$ corresponding to the Dirichlet case.)

The limiting problem for (4.46) is posed on the graph consisting of two edges adjacent in the single vertex. Let $V(s) = (v_1(s), v_2(s))^t$ be the vector of representatives of an eigenfunction of the limiting problem along the edges of the graph. Then either

\[ V(s) = (v^N_h(s), v^N_h(s))^t, \]

or

\[ V(s) = (v^D_h(s), -v^D_h(s))^t. \]

It is easy to see then that the following boundary conditions at the vertex must be imposed:

\[ V(0) + hTV'(0) = 0, \]

where

\[ T = \frac{1}{2} \begin{pmatrix} \sigma^N + \sigma^D & \sigma^N - \sigma^D \\ \sigma^N - \sigma^D & \sigma^N + \sigma^D \end{pmatrix}. \]

### 4.4 Explicit example for zero-curvature case

Let $\kappa \equiv 0$. Then the eigenvalue problem for the operator $L_h$ takes the form

\[
-\frac{d^2}{ds^2} v_h^{(k)} = \mu_h^{(k)} v_h^{(k)}, \quad s \in (0, 1),
\]

\[
v_h^{(k)}(0) + h\sigma \frac{d}{ds} v_h^{(k)}(0) = 0,
\]

\[ v_h^{(k)}(1) = 0. \]
The eigenvalues $\mu_h^{(k)}$ (except the first eigenvalue in the case $\sigma > 0$ which we do not consider here) converge to the eigenvalues of the problem

$$
-\frac{d^2}{ds^2}v_0^{(k)} = \mu_0^{(k)}v_0^{(k)}, \quad s \in (0, 1),
$$

$$
v_0^{(k)}(0) = v_0^{(k)}(1) = 0,
$$
as stated in Theorem 4.2.8. The solutions to the latter are elementary:

$$
\mu_0^{(k)} = k^2\pi^2, \quad k = 1, 2, \ldots,
$$

$$
v_0^{(k)} = \sin(k\pi s).
$$

It is easy to see that the eigenfunctions of $L_h$ are given by

$$
v_h^{(k)} = \sin\left(\sqrt{\mu_h^{(k)}}(1-s)\right),
$$

where $\mu_h^{(k)}$ can be found from the boundary condition. Namely, the eigenvalues $\mu_h^{(k)}$ are all the solutions of the equation

$$
\tan\left(\sqrt{\mu_h^{(k)}}\right) = h\sigma\sqrt{\mu_h^{(k)}}.
$$

Asymptotically one has

$$
\mu_h^{(k)} = k^2\pi^2 + h^2\sigma k^2\pi^2 + O(h^2), \quad k = 1, 2, \ldots, \sigma < 0
$$

$$
\mu_h^{(k+1)} = k^2\pi^2 + h^2\sigma k^2\pi^2 + O(h^2), \quad k = 1, 2, \ldots, \sigma > 0.
$$

These perfectly agree with the results of Chapter 3 that give

$$
v_0 = cv_h^{(k)}
$$

$$
\lambda_0 = k^2\pi^2,
$$

$$
\lambda_1 = 2\sigma k^2\pi^2,
$$

for some $k = 1, 2, \ldots$, and some coefficient $c$.

From Theorem 4.2.8 it follows that for any $k = 1, 2, \ldots$, there exist an eigenvalue $\lambda_h$ of the operator $A_h$ in $\Omega_h$ such that

$$
\lambda_h = h^{-2}\pi^2 + k^2\pi^2 + h^2\sigma k^2\pi^2 + O(h^{3/2}).
$$
Chapter 5

Localisation effects and the bottom of the spectrum

In the above two chapters we studied the eigenvalues of problem (3.2) generated by the first transversal eigenvalue $\pi^2$. Now we will address the behaviour of the bottom of the spectrum of $A_h$. The first eigenvalues of $A_h$ correspond to the so-called bound states, i.e. eigenvalues of the operator $A_\alpha$ (see (3.24)) with the corresponding eigenfunctions localised near the end of the semi-infinite strip and exponentially decaying at infinity. The related eigenfunctions of $A_h$ demonstrate the same behaviour: they are confined to the slanted end of $\Omega_h$ and decay at the rate of order $\exp(-h^{-1}lx_1)$, where $l > 0$ is some fixed number. Thus, our purpose is to study the point spectrum of the operator $A_\alpha$ lying below its essential spectrum $[\pi^2, \infty)$. In the first section we develop some methods for estimation of the number of bound states. The second section is devoted to the proof of the monotonicity of the first bound state as a function of the angle of the slant.

5.1 Boundary localisation

Theorem 5.1.1. There exists at least one eigenvalue $\lambda$ of the operator $A_\alpha$ such that $\lambda < \pi^2$, the corresponding eigenfunction $u(y)$ decays exponentially at infinity ($u(y) \sim \exp(-ly_1)$ as $y_1 \to \infty$, where $l$ is some positive number). Then there exists $h_0 > 0$ and a constant $C$ independent of $h$ such that for any $0 < h \leq h_0$ there exist an eigenvalue $\lambda_h$ of the operator $A_h$ such that

$$|\lambda_h - h^{-2}\lambda| \leq Ce^{-lh^{-1}}.$$
Moreover, a function
\[ u_h^{\text{appr}} = \chi \left( \frac{2x_1}{s_0} \right) u(h^{-1}x) \]
approximates eigenfunctions of \( A_h \) in the following sense: for any \( d > 0 \) and any \( 0 < h \leq h_0 \) there exist coefficients \( c_i(h) \) such that
\[
\left\| u_h^{\text{appr}} - \sum_{|\lambda_{i,h} - h^{-2}\lambda| \leq d} c_i(h) u_{i,h} \right\|_{L^2(\Omega_h)} \leq C d^{-1} e^{-lh^{-1}}.
\]

**Proof.** It is well known, see e.g. [69], that there exists at least one eigenvalue of \( A_\alpha \) below its essential spectrum and the corresponding eigenfunction decays exponentially at infinity. The rest of the proof is analogous to the proof of Theorem 3.5.1 although is much simpler. One only needs to notice that the discrepancy \( -\Delta x u_h^{\text{appr}} - h^{-2}\lambda u_h^{\text{appr}} \) is of order \( e^{-lh^{-1}} \).

In view of Theorem 5.1.1 it is important to describe somehow the discrete spectrum of \( A_\alpha \). Our aim is to study the relation between the value of angle \( \alpha \) and the behaviour of the discrete spectrum of \( A_\alpha \). For this purpose we will use methods from [62, Chapter IV]. Let
\[
\Psi_0 := \{(y_1, y_2) \mid 0 < y_1 < \cot(\alpha), 0 < y_2 < \tan(\alpha) y_1\},
\Psi_1 := \{(y_1, y_2) \mid y_1 > \cot(\alpha), y_2 \in (0, 1)\}.
\]

Denote by \( T_0 \) a self-adjoint operator for the negative Laplacian \( -\Delta \) in \( \Psi_0 \) with Dirichlet boundary condition on \( y_2 = 0 \) and Neumann boundary condition on the rest of the boundary of \( \Psi_0 \). Analogously, we denote by \( T_1 \) a self-adjoint operator for \( -\Delta \) in \( \Psi_1 \) with Dirichlet boundary condition on \( y_2 = 0, y_2 = \pi \) and Neumann boundary condition on the rest of the boundary of \( \Psi_1 \). Denote by
$D_0$ and $D_1$ spaces of functions from $H^1(\Psi_0)$ and $H^1(\Psi_1)$ vanishing on $y_2 = 0$ and $y_2 = 0, y_2 = \pi$ respectively. It is well known that $\sigma(T_1) = [\lambda_2, \infty)$. The following lemma provides a tool of estimating the number of eigenvalues of $A_\alpha$ from above (cf. [62]).

**Lemma 5.1.2.** The number of eigenvalues of the operator $A_\alpha$ lying below its essential spectrum can be estimated from above by the number of eigenvalues of $T_0$ lying below $\sigma_{\text{ess}}(A_\alpha)$.

**Proof.** Let $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_k < \pi^2$ and $\omega_1 \leq \omega_2 \leq ... \leq \omega_n < \pi^2$ be all eigenvalues of $A_\alpha$ and $T_0$ respectively and $v_1, ..., v_k, \psi_1, ..., \psi_n$ be the corresponding orthonormalised eigenfunctions. Our aim is to prove that

$$k \leq n. \tag{5.2}$$

Assume the contrary, i.e. that $k > n$. Then there exists a linear combination of eigenfunctions of $A_\alpha$

$$v = \sum_{i=1}^{k} c_i v_i,$$

such that

$$\int_{\Psi_0} v \psi_j \, dx = 0, \quad j = 1, \ldots, n.$$

For each eigenfunction of $A_\alpha$ we have

$$\|\nabla v_i\|_{L^2(\Pi_\alpha)}^2 = \gamma_i \|v_i\|_{L^2(\Pi_\alpha)}^2 = \gamma_i.$$

Then we obtain the following inequality for $v$:

$$\|\nabla v\|_{L^2(\Pi_\alpha)}^2 = \sum_{i=1}^{k} c_i^2 \|\nabla v_i\|_{L^2(\Pi_\alpha)}^2 = \sum_{i=1}^{k} c_i^2 \gamma_i \|v_i\|_{L^2(\Pi_\alpha)}^2 \leq \gamma_k \|v\|_{L^2(\Pi_\alpha)}^2 < \pi^2 \|v\|_{L^2(\Pi_\alpha)}^2. \tag{5.3}$$

The lower bound of the spectrum of $T_1$ is $\pi^2$, so by the variational principle

$$\|\nabla v\|_{L^2(\Psi_1)}^2 \geq \pi^2 \|v\|_{L^2(\Psi_1)}^2.$$
Deducting the last inequality from (5.3) we obtain
\[ \|\nabla v\|^2_{L^2(\Psi_0)} < \pi^2 \|v\|^2_{L^2(\Psi_0)}. \]

Then, since \( v \in D_0 \), one should have
\[
\inf_{D_0 \ni \psi_j \neq 0} \frac{\|\nabla \psi\|^2_{L^2(\Psi_0)}}{\|\psi\|^2_{L^2(\Psi_0)}} \leq \frac{\|\nabla v\|^2_{L^2(\Psi_0)}}{\|v\|^2_{L^2(\Psi_0)}} < \pi^2,
\]
which is impossible because there are only \( n \) eigenvalues of \( T_0 \) less than \( \pi^2 \). We arrive at a contradiction. Hence \( k \leq n \).

**Theorem 5.1.3.** If \( \alpha = \pi/4 \) the operator \( A_\alpha \) has exactly one eigenvalue lying below its essential spectrum.

**Proof.** Let \( \alpha \) be equal to \( \pi/4 \), so that \( \Psi_0 \) is an isosceles right-angled triangle. Let \( \psi \) be a solution to the eigenvalue problem
\[
-\Delta \psi = \omega \psi, \ y \in \Psi_0,
\]
\[
\psi = 0, \ y_2 = 0;
\]
\[
\frac{\partial \psi}{\partial \nu} = 0, \ y_1 = 1 \text{ or } y_2 = y_1.
\]

(5.4)

We can extend \( \psi \) to the unit square \((0,1)^2\) symmetrically reflecting it with respect to the line \( y_1 = y_2 \), i.e. by the formula

for any \((y_1, y_2) \in (0,1)^2 \setminus \Pi_0, \ \psi(y_1, y_2) = \psi(y_2, y_1), \ (y_2, y_1) \in \Psi_0.\)

Then, by the symmetry principal for the Laplacian, the extended function, which we still denote by \( \psi \), is a solution to the eigenvalue problem
\[
-\Delta \psi = \omega \psi, \ y \in (0,1)^2,
\]
\[
\psi = 0, \ y_1 = 0 \text{ or } y_2 = 0,
\]
\[
\frac{\partial \psi}{\partial \nu} = 0, \ y_1 = 1 \text{ or } y_2 = 1.
\]

(5.5)

So we can seek the solution to (5.4) amongst the eigenfunctions of the operator given by (5.5). By separation of variables, all the eigenfunctions and correspond-
The eigenvalues of \((5.5)\) are the following:

\[
\psi_{n,m} = \sin \left( \pi \left( 2n - 1 \right) y_1 / 2 \right) \sin \left( \pi \left( 2m - 1 \right) y_2 / 2 \right),
\]

\[
\omega_{n,m} = \frac{\pi^2}{4} \left( (2n - 1)^2 + (2m - 1)^2 \right), \ n, m \in \mathbb{N}.
\]

It is easy to see that \(\omega_1 = \omega_{1,1} = \frac{\pi^2}{4}\) and \(\psi_1 = \psi_{1,1}\) are eigenvalue and eigenfunction of \((5.4)\). The next eigenvalue of the problem \(\omega_2\) is greater than \(\frac{\pi^2}{4}\) (since \(\omega_2 \geq \omega_{1,2} = \omega_{2,1} = \frac{5}{4} \pi^2\)). Then the theorem follows from Theorem \(5.1.1\) and Lemma \(5.1.2\).

Now, when we have a method of estimating the number of eigenvalues \(A_\alpha\) from above, we would like to know how to estimate it from below. This can be done in somewhat similar way. Let \(\Psi_2\) be a bounded domain contained between the lines \(y_2 = 0, \ y_2 = 1, \ y_2 = \tan(\alpha) y_1\) and some smooth simple curve \(\Gamma^3\) lying entirely in \(\Pi_\alpha\) such that its one end belongs to \(\{ y_1 \geq 0, \ y_2 = 0 \}\), another belongs to \(\{ y_1 \geq \cot(\alpha), \ y_2 = 1 \}\) and it has no other common points with \(\partial \Pi_\alpha\), see Figure 5-2. Let \(T_2\) be an operator defined by \(-\Delta\) in \(\Psi_2\) with Dirichlet boundary conditions imposed on \(\partial \Psi_2 \cap \Gamma^2_\alpha \cup \Gamma^3\) and Neumann conditions on the rest of the boundary (i.e. on \(\Gamma^1_\alpha\)). Denote by \(D_2\) the space of functions from \(H^1(\Psi_2)\) vanishing on \(\partial \Psi_2 \cap \Gamma^2_\alpha \cup \Gamma^3\). Then the following assertion is true (cf. [62]).

**Lemma 5.1.4.** The number of eigenvalues of the operator \(A_\alpha\) lying below its essential spectrum can be estimated from below by the number of eigenvalues of \(T_2\) lying below \(\sigma_{\text{ess}}(A_\alpha)\).

**Proof.** Denote all the eigenvalues lying below \(\sigma_{\text{ess}}(A_\alpha)\) and the corresponding eigenfunctions of \(T_2\) by \(\mu_i\) and \(\varphi_i, \ i = 1, \ldots, m\), respectively, \(\mu_1 \leq \ldots \leq \mu_m < \)
Let \( k \) be the number of eigenvalues of \( A_\alpha \) as in Lemma 5.1.2. Assume to the contrary that \( m > k \). Then there exists a linear combination of eigenfunctions of \( T_2 \),

\[
\varphi = \sum_{i=1}^{m} c_i \varphi_i,
\]

orthogonal to all \( v_i, i = 1, \ldots, k \). Notice that due to the boundary conditions for \( \varphi \) we can extend it by zero to the rest of \( \Pi_\alpha \), so that the extension \( \varphi \in H_0^1(\Pi_\alpha, \Gamma^2_\alpha) \), where \( H_0^1(\Pi_\alpha, \Gamma^2_\alpha) \) denotes the space of functions from \( H^1(\Pi_\alpha) \) vanishing on \( \Gamma^2_\alpha \), see (3.24). Then we have

\[
\inf_{H_0^1(\Pi_\alpha, \Gamma^2_\alpha)} \sup_{v \neq v_j} \frac{||\nabla v||_{L^2(\Pi_\alpha)}}{||v||_{L^2(\Pi_\alpha,0)}} \leq \frac{||\nabla \varphi||_{L^2(\Pi_\alpha)}}{||\varphi||_{L^2(\Pi_\alpha)}} \leq \mu_m < \pi^2,
\]

which means existence of \( k + 1 \) eigenvalues of \( A_\alpha \) below its essential spectrum. We obtain a contradiction. 

\[\square\]

**Remark 5.1.5.** The argument of Lemmas 5.1.2 and 5.1.4 is known as Dirichlet-Neumann bracketing.

Suppose now that \( \Psi_2 \) is a sector given in the polar coordinates by

\[
\Psi_2 := \{ (\rho, \theta)| 0 < \rho < (\sin(\alpha))^{-1}, \ 0 < \theta < \alpha \},
\]

see Figure 5-3. Let us consider an eigenvalue problem

\[
-\Delta \varphi = \mu \varphi, \ y \in \Psi_2,
\]

\[
\varphi = 0, \ \theta = 0 \text{ or } \rho = (\sin(\alpha))^{-1},
\]

\[
\frac{\partial \varphi}{\partial \theta} = 0, \ \theta = \alpha.
\]

Figure 5-3: Semi-infinite cylinder

\[\pi^2.\]
The Laplace operator in the polar coordinates is given by the formula

\[ \Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2}. \]

We can separate the variables in the equation, let \( \varphi = u(\rho)v(\theta) \). Then we obtain the following equations for \( v \) and \( u \):

\[
-v'' = \nu^2 v, \\
v(0) = 0, \quad v'(\alpha) = 0; \\
u'' + \frac{1}{\rho}u' + \left( \frac{\mu - \nu^2}{\rho^2} \right) u = 0, \\
u(0) = u \left( (\sin(\alpha))^{-1} \right) = 0.
\] (5.7)

Obviously, the sequence of eigenfunctions and eigenvalues satisfying the first equation is given by

\[
v_i = \sin \left( \frac{\pi}{2\alpha} (2i - 1) \theta \right), \quad i = 1, 2, \ldots, \\
\nu_i = \frac{\pi}{2\alpha} (2i - 1).
\]

Making the substitution \( r = \sqrt{\mu \rho} \), \( u(\rho) = \tilde{u}(r) \) in the second equation one arrives at Bessel equation

\[
\tilde{u}'' + \frac{1}{r} \tilde{u}' + \left( 1 - \frac{\nu^2}{r^2} \right) \tilde{u} = 0.
\]

The solution to this equation is given by Bessel function

\[
J_{\nu_i}(r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu_i + k + 1)} \left( \frac{r}{2} \right)^{\nu_i + 2k}.
\]

Denote by \( j_{\nu_i,l}, l, j = 1, 2, \ldots \) zeros of Bessel function \( J_{\nu_i} \). In order to satisfy boundary conditions in (5.7) we need to make a substitution \( r = \sqrt{\mu_{i,l} \rho} \) where

\[
\mu_{i,l} = \sin^2(\alpha) j_{\nu_i,l}^2, \quad i, l = 1, 2, \ldots.
\] (5.8)

Then the function

\[
u_{i,l} = J_{\nu_i}(\sqrt{\mu_{i,l} \rho})
\]
Solves problem (5.7) with \( \mu = \mu_{i,l} \). Then the solution to eigenvalue problem (5.6) is given by the sequence of eigenvalues (5.8) and corresponding eigenfunctions

\[
\psi_{i,l} = \sin \left( \frac{\pi}{2\alpha} (2i - 1) \theta \right) J_{\nu_i}(\sqrt{\mu_{i,l}} \rho), \quad i, l = 1, 2, \ldots.
\]

Thus we have proved the following

**Theorem 5.1.6.** The number of eigenvalues of \( A_\alpha \) lying below its essential spectrum is greater than or equal to the number of those eigenvalues

\[
\mu_{i,l} = \sin^2(\alpha) j_{\nu_i,l}^2, \quad i, l = 1, 2, \ldots
\]

of problem (5.6) which are less than \( \pi^2 \), where \( j_{\nu,1} < j_{\nu,2} < \ldots \) are zeros of Bessel function \( J_{\nu} \) and

\[
\nu_i = \frac{\pi}{2\alpha} (2i - 1).
\]

Notice that for the first zero of Bessel function \( J_{\nu} \) the following inequality is valid:

\[
j_{\nu,1} > \sqrt{\nu(\nu + 2)},
\]

see e.g. [81]. Hence we have

\[
\mu_{i,1} > \sin^2(\alpha) \left( \frac{(2i - 1)^2}{4\alpha^2} \pi^2 + \frac{2i - 1}{\alpha} \pi \right).
\]

The function \( \sin(\alpha)/\alpha \) is a strictly decreasing function on the interval \( \alpha \in (0, \pi/2) \). Then it is easy to see by direct calculations that

\[
\mu_{i,1} > \pi^2, \quad i \geq 2.
\]

On the other hand the asymptotic formula for zeros of Bessel function \( J_{\nu} \) for large values of \( \nu \),

\[
j_{\nu,l} = \nu + o(\nu), \quad l \in \mathbb{N},
\]

see e.g. [51], [65], ensures that for small enough \( \alpha > 0 \) there are arbitrary many eigenvalues of problem (5.6) lying below \( \pi^2 \). More precisely we have

\[
\mu_{1,l} = \frac{\pi^2}{4} + o(1), \quad l \in \mathbb{N},
\]

as \( \alpha \to 0 \). Then the following assertion follows from Theorem 5.1.6.
Theorem 5.1.7. For small enough $\alpha > 0$ there are arbitrary many eigenvalues of $A_\alpha$ lying below its essential spectrum.

5.2 Monotonicity of the first eigenvalue with respect to the angle $\alpha$

It is clear that the first eigenvalue of problem (3.24) is simple. It is interesting question how it depends on the value of the angle $\alpha$. Bearing in mind the results of the previous section it is natural to conjecture that the first eigenvalue decays as $\alpha$ gets smaller.

Theorem 5.2.1. Let $\lambda_\alpha$ be the first eigenvalue of $A_\alpha$. Then $\lambda_\alpha$ is a strictly increasing function of the argument $\alpha \in (0, \pi/2)$.

Proof. First let us consider eigenvalue problem (3.24) in the sequence of domains $\Pi_{\alpha_i}$, $i = 1, \ldots, n$, corresponding to angles $\alpha_i$, where $0 < \alpha_1 < \ldots < \alpha_n \leq \pi/2$, $\alpha_i = \alpha_1 + (i-1)\Delta \alpha$, $\Delta \alpha > 0$, $i = 2, \ldots, n$, and $n$ is such that $\alpha_{n+1} > \pi/2$. Denote by $\lambda_{\alpha_i} < \pi^2$ and $\varphi_{\alpha_i}$, first eigenvalues and respective eigenfunctions of the corresponding problems. A position of a domain on the coordinate plane is insignificant, therefore for the sake of simplified and more illustrative narration we shift the domains $\Pi_{\alpha_i}$ by $\cot(\alpha_1) - \cot(\alpha_i)$ in positive direction of axis $x$ so as the top vertices of $\Pi_{\alpha_1}$ and $\Pi_{\alpha_i}$ to coincide (in the point $C$), see Figure 5-4. Points $B$ and $D$ on the Figure 5-4 are located such that triangles $\Delta ABC$ and $\Delta ADC$ are equal with sides $AB = AD$ and $BC = CD$. We denote the domains enclosed within $\Delta ABC$ and $\Delta ADC$ by $\Sigma_1$ and $\Sigma_2$ respectively.

The following equality is true:

$$\| \nabla \varphi_{\alpha_2} \|_{L^2(\Pi_{\alpha_2})}^2 = \lambda_{\alpha_2} \| \varphi_{\alpha_2} \|_{L^2(\Pi_{\alpha_2})}^2. \quad (5.9)$$

By the variational properties of eigenvalues

$$\lambda_{\alpha_1} = \inf_{f \in H^1_0(\Pi_{\alpha_1}; \Gamma_{\alpha_1}^2)} \frac{\| \nabla f \|_{L^2(\Pi_{\alpha_1})}^2}{\| f \|_{L^2(\Pi_{\alpha_1})}^2}. \quad (5.10)$$

Let us extend the eigenfunction $\varphi_{\alpha_2}$ into $\Sigma_1$ by symmetric reflection against the line $AC$ and by zero into $\Delta OAB$. The extension, which we denote by $\tilde{\varphi}_{\alpha_2}$, belongs to $H^1_0(\Pi_{\alpha_1}; \Gamma_{\alpha_1}^2)$ due to the boundary conditions imposed on $\varphi_{\alpha_2}$. Thus
we have
\[ \lambda_{\alpha_1} \leq \frac{\| \nabla \tilde{\varphi}_{\alpha_2} \|^2_{L^2(\Pi_{\alpha_1})}}{\| \tilde{\varphi}_{\alpha_2} \|^2_{L^2(\Pi_{\alpha_1})}}. \] (5.11)

Notice that from the symmetry of \( \tilde{\varphi}_{\alpha_2} \) against the line \( AC \) follows that \( \| \nabla \tilde{\varphi}_{\alpha_2} \|_{L^2(\Sigma_1)} = \| \nabla \varphi_{\alpha_2} \|_{L^2(\Sigma_1)} = \| \varphi_{\alpha_2} \|_{L^2(\Sigma_2)} \). Then we can write (5.11) as
\[
\| \nabla \varphi_{\alpha_2} \|^2_{L^2(\Pi_{\alpha_2})} + \| \nabla \varphi_{\alpha_2} \|^2_{L^2(\Sigma_2)} \geq \lambda_{\alpha_1} \left( \| \varphi_{\alpha_2} \|^2_{L^2(\Pi_{\alpha_2})} + \| \varphi_{\alpha_2} \|^2_{L^2(\Sigma_2)} \right). \] (5.12)

Deducing (5.9) from (5.12) we obtain
\[
\| \nabla \varphi_{\alpha_2} \|^2_{L^2(\Sigma_2)} \geq (\lambda_{\alpha_1} - \lambda_{\alpha_2}) \| \varphi_{\alpha_2} \|^2_{L^2(\Pi_{\alpha_2})} + \lambda_{\alpha_1} \| \varphi_{\alpha_2} \|^2_{L^2(\Sigma_2)}.
\]

Now we deduct the latter from (5.9):
\[
\| \nabla \varphi_{\alpha_2} \|^2_{L^2(\Pi_{\alpha_2})} = \| \nabla \varphi_{\alpha_2} \|^2_{L^2(\Pi_{\alpha_2})} - \| \nabla \varphi_{\alpha_2} \|^2_{L^2(\Sigma_2)} < \]
\[
< (2\lambda_{\alpha_2} - \lambda_{\alpha_1}) \| \varphi_{\alpha_2} \|^2_{L^2(\Pi_{\alpha_2})} + 2(\lambda_{\alpha_2} - \lambda_{\alpha_1}) \| \varphi_{\alpha_2} \|^2_{L^2(\Sigma_2)}. \] (5.13)

Assume that \( \lambda_{\alpha_1} \geq \lambda_{\alpha_2} \) and denote \( \Delta \lambda = \lambda_{\alpha_1} - \lambda_{\alpha_2} \geq 0 \). Since \( \varphi_{\alpha_2} \in H^1_0(\Pi_{\alpha_3}, \Gamma^2_{\alpha_3}) \) we conclude from (5.13) that
\[
\lambda_{\alpha_3} = \inf_{f \in H^1_0(\Pi_{\alpha_3}, \Gamma^2_{\alpha_3})} \frac{\| \nabla f \|^2_{L^2(\Pi_{\alpha_3})}}{\| f \|^2_{L^2(\Pi_{\alpha_3})}} \leq \frac{\| \nabla \varphi_{\alpha_2} \|^2_{L^2(\Pi_{\alpha_3})}}{\| \varphi_{\alpha_2} \|^2_{L^2(\Pi_{\alpha_3})}} \leq \lambda_{\alpha_2} - \Delta \lambda.
\]

Then we obtain \( \lambda_{\alpha_2} - \lambda_{\alpha_3} \geq \Delta \lambda = \lambda_{\alpha_1} - \lambda_{\alpha_2} \). Reasoning by induction we derive
that

$$\lambda_{\alpha n} \leq \lambda_{\alpha 1} - (n - 1)\Delta \lambda < \pi^2. \tag{5.14}$$

Now let us prove the monotonicity property of $\lambda_\alpha$. We aim to show that the first eigenvalue $\lambda_\alpha$ of (3.24) is strictly increasing function of the argument $\alpha \in (0, \pi/2)$. Reasoning by contradiction we assume that there exist $\alpha'$ and $\alpha''$ from $(0, \pi/2)$ such that $\alpha' < \alpha'' < \pi/2$ and $\lambda_{\alpha'} \geq \lambda_{\alpha''}$. It is obvious that in this case one can choose $\alpha \in [\alpha', \alpha'']$ such that for any $\delta > 0$ there exists $0 < \Delta \alpha < \delta$ satisfying $\lambda_\alpha \geq \lambda_{\alpha + \Delta \alpha}$. Then by (5.14) we have

$$\lambda_\gamma < \lambda_\alpha < \pi^2, \tag{5.15}$$

where the angle $\gamma = \alpha + n\Delta \alpha$ for some $n$ such that $\gamma \leq \pi/2$, $|\gamma - \pi/2| < \Delta \alpha$. So $\gamma$ can be chosen arbitrary close to $\pi/2$.

On the other hand the eigenvalue $\lambda_\gamma$ must be close to the bottom of the essential spectrum of problem (3.24) when the angle $\gamma$ is close to $\pi/2$. Indeed, one can show this using Poincaré inequality. Let us introduce a new coordinates obtained from $(y_1, y_2)$ by rotation on the angle $-(\pi/2 - \gamma)$:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} \cos(\pi/2 - \gamma) & - \sin(\pi/2 - \gamma) \\ \sin(\pi/2 - \gamma) & \cos(\pi/2 - \gamma) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

so that the part of the boundary $\Gamma_1^{\gamma}$ lies on the positive part of the axis $y_2'$. We denote by $\Pi_\gamma'$ the domain $\Pi_\gamma$ in the new coordinates:

$$\Pi_\gamma' = \left\{ (y_1', y_2') \left| y_1' > 0, y_1' \tan\left(\frac{\pi}{2} - \gamma\right) < y_2' < y_1' \tan\left(\frac{\pi}{2} - \gamma\right) + \frac{1}{\sin(\gamma)} \right. \right\}.$$

For a function $\varphi \in H_0^1(0, 1/\sin(\gamma))$ we have well known Poincaré inequality with an explicit constant,

$$\int_0^{1/\sin(\gamma)} (\varphi(y_2'))^2 dy_2' \leq (\pi \sin(\gamma))^{-2} \int_0^{1/\sin(\gamma)} \left( \frac{d}{dy_2'} \varphi(y_2') \right)^2 dy_2'.$$

Due to the properties of rotation the modulus of the gradient of function remains
the same and the Jacobian equals 1. So we derive for $\varphi \in H_0^1(\Pi_{\gamma}, \Gamma_{\gamma}^2)$

$$
\int_{\Pi_{\gamma}} \varphi^2 dx dy = \int_{\Pi_{\gamma}} \varphi^2 dy'_1 dy'_2 \leq (\pi \sin(\gamma))^{-2} \int_{\Pi_{\gamma}} \left( \frac{d}{dy'_2} \varphi \right)^2 dy'_1 dy'_2 \leq (\pi \sin(\gamma))^{-2} \int_{\Pi_{\gamma}} |\nabla' \varphi|^2 dy'_1 dy'_2 = (\pi \sin(\gamma))^{-2} \int_{\Pi_{\gamma}} |\nabla \varphi|^2 dx dy.
$$

Thus, as $\gamma$ tends to $\pi/2$ the first eigenvalue

$$
\lambda_{\gamma} = \inf_{\varphi \in H_0^1(\Pi_{\gamma}, \Gamma_{\gamma}^2)} \frac{\|\nabla \varphi\|^2_{L^2(\Pi_{\gamma})}}{\|\varphi\|^2_{L^2(\Pi_{\gamma})}} \geq (\pi \sin(\gamma))^2
$$

tends to $\pi^2$, which contradicts to (5.15). This proves the theorem. \qed
Bibliography


