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Tensors of Comodels and Models for Operational Semantics

Gordon Plotkin [1]

School of Informatics, University of Edinburgh, Edinburgh EH9 3JZ, UK

John Power [2]

Department of Computer Science, University of Bath, Bath, BA2 7AY, UK

Abstract

In seeking a unified study of computational effects, in particular in order to give a general operational semantics agreeing with the standard one for state, one must take account of the coalgebraic structure of state. Axiomatically, one needs a countable Lawvere theory L, a comodel C, typically the final one, and a model M, typically free; one then seeks a tensor $C \otimes M$ of the comodel with the model that allows operations to flow between the two. We describe such a tensor implicit in the abstract category theoretic literature, explain its significance for computational effects, and calculate it in leading classes of examples, primarily involving state.

Key words: Countable Lawvere theory, model, comodel, global state, arrays, free cocompletion, tensor.

1 Introduction

Over the past decade, in collaboration with a number of other researchers, and following Eugenio Moggi's seminal monadic approach to notions of computation [10], we have been developing an algebraic theory of computational effects. This theory emphasises the operations that give rise to the effects at hand, and the equations that hold between them: see [14] for an overview.

One goal of this project was to give an axiomatic account of the various methods of combining effects [3,4,5]. Another, indeed the focus of the first paper of the series [11], was to develop a unified theory of structural operational semantics for effects; unfortunately, however, the axiomatics of [11] had the

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severe limitation of not accounting for the example of state, or for any combination of effects including state. More recently, we have begun to appreciate the importance of coalgebra in understanding the dynamics of state [16] and in this paper we start to rectify the situation by developing the combination of algebra and coalgebra which we believe will be needed.

In modelling state, one typically has assignment and dereferencing constructors, := and !, with typing rules of the form:

$$\frac{M : \mathbf{loc} \quad N : \mathbf{val}}{(M := N) : 1} \qquad\qquad \frac{M : \mathbf{loc}}{!M : \mathbf{val}}$$

where **loc** is a type to be modelled by a finite set of locations Loc and **val** is the type of values, modelled by a countable non-empty set V. The structural operational semantics of state typically involves transition systems of the form:

$$\langle s, M \rangle \longrightarrow \langle s', M' \rangle \tag{1}$$

where M and M' are closed terms of the same type, σ say, and s and s' are states, i.e., elements of $S =_{\mathrm{def}} V^{\mathrm{Loc}}$. (We reverse the usual order to fit in better with the tensor introduced below.)

The transitions can be generated by such rules as:

$$\langle s, E[l := v] \rangle \longrightarrow \langle s[v/l], E[*] \rangle \tag{2}$$

$$\langle s, E[!l] \rangle \longrightarrow \langle s, E[s(l)] \rangle \tag{3}$$

where $E[\ ]$ is an evaluation context. An adequate denotational semantics must, in some way, identify the two sides of these transitions. For example, using Moggi's state monad $T_S =_{\mathrm{def}} (S \times -)^S$ the denotation $[\![M]\!]$ of M of type σ is in $T_S([\![\sigma]\!])$, where $[\![\sigma]\!]$ is the denotation of σ, and one has:

$$[\![M]\!](s) = [\![M']\!](s') \tag{4}$$

However, applying the general operational semantics of [11] to the case of state yields transitions applied to a term M only, with no state parameter.

Analysis of equation (4) in the cases of rules (2) and (3) shows it has a specific form. First, assignment and dereferencing are modelled by evident maps

$$g_{:=} : \mathrm{Loc} \times V \longrightarrow T_S(1) \qquad\qquad g_! : \mathrm{Loc} \longrightarrow T_S(V)$$

Now, as we recall in detail in Section 2, the countable Lawvere theory L_S for global state is generated by two operations

$$a : 1 \longrightarrow \mathrm{Loc} \times V \qquad\qquad d : V \longrightarrow \mathrm{Loc}$$

subject to seven equations [12,16]. These operations yield two *algebraic operations*, viz families of maps, natural in the Kleisli category

$$a_{T_S(X)} : T_S(X) \longrightarrow T_S(X)^{\mathrm{Loc} \times V} \qquad\qquad d_{T_S(X)} : T_S(X)^V \longrightarrow T_S(X)^{\mathrm{Loc}}$$

and the maps $g_{:=}$ and $g_!$ modelling assignment and dereferencing appear as the corresponding *generic effects* [13].

Next, as we recall in Section 3, a comodel of L_S, i.e., a countable coproduct preserving functor from L_S^{op} to Set, amounts to a set Y together with *co-operations*

$$a_Y : (\mathrm{Loc} \times V) \times Y \longrightarrow Y \qquad d_Y : \mathrm{Loc} \times Y \longrightarrow V \times Y$$

subject, appropriately interpreted, to the same seven equations; the category of such coalgebras is, in turn, equivalent to the category of arrays in the sense of [16]. Modulo a transposition, the generic effects provide the final such coalgebra on the set of states S.

We can now consider equation (4), say in the case of Rule (2). One can show that $[\![E[l := v]]\!] = a_{T([\![\sigma]\!])}((l,v), [\![E[*]]\!])$ and with this the equation becomes:

$$a_{T([\![\sigma]\!])}((l,v), [\![E[*]]\!])(s) = [\![E[*]]\!](a_S((l,v), s))$$

We see that the equation swaps the coalgebra map a_S for the algebra map $a_{T([\![\sigma]\!])}$. The same holds, albeit a little less obviously, for rule (3) with respect to d_S and $d_{T([\![\sigma]\!])}$. This swapping of an algebra map with a coalgebra map is characteristic of the maps $\otimes$ involved in tensors, for example the tensor of two bimodules. In our case $\otimes$ is application, so we would write the above equation as:

$$s \otimes a_{T([\![\sigma]\!])}((l,v), [\![E[*]]\!]) = a_S((l,v), s) \otimes [\![E[*]]\!]$$

with the idea that:

$$S \otimes T_S([\![\sigma]\!]) = S \times [\![\sigma]\!]$$

In general, given an algebra A and coalgebra C we seek the universal map:

$$C \times A \xrightarrow{\otimes} C \otimes A$$

such that for every operation $f : I \to J$, we have:

$$s \otimes f_A(\gamma)(j) = s' \otimes \gamma(i)$$

for every $s \in C$, $\gamma \in A^I$ and $j \in J$, where $\langle i, s' \rangle = f_C(j, s)$. Axiomatically this involves a countable Lawvere theory L, a comodel $C : L \longrightarrow Set$, and a model $M : L^{op} \longrightarrow Set$. It turns out that the tensor can be equivalently seen as factoring the sum of all pairs $Ca \times Ma$ by allowing the swapping of the image Cf of a map f in L with the image Mf of f, while respecting the countable product structure of the countable Lawvere theory L. The tensor is constructed using a countable coproduct-respecting variant of the theorem that the free cocompletion of a small category D is given by the Yoneda embedding $\mathrm{Y} : D \longrightarrow [D^{op}, Set]$.

We explore the relevant abstract mathematics in Section 4 and show that for an arbitrary L-comodel C, the tensor $C \otimes T_L(X)$ of C with the free L-model on a set X is $C1 \times X$. This result confirms and generalises the above informal discussion of state; it also applies to read-only state in combination with other effects. In Section 5, we give general results allowing the calculation of the tensor $C \otimes M$ in two other cases: the combination of global state or monoid actions with other effects. But what about a combination of comodels, for instance having both read-only and global state? In order to account for these, in Section 6 we describe an operation $C \circ C'$ on comodels and give a formula for $(C \circ C') \otimes M$ in terms of formulae involving C and C' individually. Appendix A gives a bicategorical view of the tensor: while we have no application for it yet, it is mathematically very natural.

We have expressed ourselves here, and we continue to express ourselves through the course of the paper, in terms of ordinary countable Lawvere theories, and models and comodels in *Set*. One would wish everything to generalise to Lawvere V-theories [15] for those V that are locally countably presentable as cartesian closed categories, such as ωCpo. For the most part it does, and we comment as appropriate. Example 6.3 and succeeding consist of two calculations of pushouts in *Set* and we do not know a suitable generalisation. Fortunately, however, the two calculations go through for ωCpo and so we can account for recursion, see [3,4], including the lifting monad among the effects not involving comodels. We also employ a result of [9] which may not enrich; however we only use it to add motivation to results that do enrich: see Theorem 5.1 and Corollary 5.2 and the discussions thereafter.

2 Models of Countable Lawvere Theories

In this section, we briefly recall the definitions of countable Lawvere theory and model, focusing on the countable Lawvere theory for global state; details are implicit in [12] and explicit in [16]. Let $\aleph_1$ denote a skeleton of the full subcategory of *Set* of the countable sets. So, up to equivalence, it contains one object n for each natural number, together with an object $\aleph_0$ to represent a countable set. It has countable coproducts.

Definition 2.1 A *countable Lawvere theory* consists of a category L with countable products and an identity-on-objects strict countable product preserving functor $J : \aleph_1^{op} \longrightarrow L$. A *model* of L in a category C with countable products is a countable product preserving functor from L to C.

The models of L in a category C with countable products form a category $Mod(L, C)$, whose arrows are given by all natural transformations.

Theorem 2.2 *For any countable Lawvere theory L and any locally presentable category C, the forgetful functor $U : Mod(L, C) \longrightarrow C$ exhibits $Mod(L, C)$ as monadic over C. In the case that $C = Set$, the induced monads T_L are precisely the countably presentable monads on Set.*

For global state, we assume we are given a non-empty finite set Loc of locations and a countable non-empty set V of values. We identify Loc with the natural number given by its cardinality, and V with a natural number or

$\aleph_0$, as appropriate (and similarly in the other examples considered below).

Definition 2.3 The countable Lawvere theory L_S for global state is the theory freely generated by maps

$$d : V \longrightarrow \mathrm{Loc} \qquad a : 1 \longrightarrow \mathrm{Loc} \times V$$

subject to the commutativity of seven diagrams, expressible as equations between infinitary terms as follows:

(i) $d_l((a_{l,v}(x))_v) = x$

(ii) $d_l(\,(d_l(\,(x_{vv'})_v\,))_{v'}\,) = d_l((x_{vv})_v)$

(iii) $a_{l,v}(a_{l,v'}(x)) = a_{l,v'}(x)$

(iv) $a_{l,v}(d_l(\,(x_v)_v)\,) = a_{l,v}(x_v)$

(v) $d_l(\,(d_{l'}(\,(x_{vv'})_{v'}\,))_v) = d_{l'}(\,(d_l(\,(x_{vv'})_v\,))_{v'})$ where $l \neq l'$

(vi) $a_{l,v}(a_{l',v'}(x)) = a_{l',v'}(a_{l,v}(x))$ where $l \neq l'$

(vii) $a_{l,v}(d_{l'}(\,(x_{v'})_{v'}\,)) = d_{l'}(\,(a_{l,v}(x_{v'}))_{v'}\,)$ where $l \neq l'$.

An equivalent version of the equations in terms of commutative diagrams appears in [16]. The definition implies an equation (viii):

$$d_l((x)_v) = x$$

The following theorem, stated in slightly different but equivalent terms, is the first main theorem of [12].

Theorem 2.4 *For any category C with countable products and countable coproducts, the forgetful functor $U : Mod(L_S, C) \longrightarrow C$ exhibits the category $Mod(L_S, C)$ as monadic over C, with monad $(S \otimes -)^S$, where S is the set V^{Loc} and $S \otimes X$ is the coproduct of S copies of X.*

Theorem 2.4 explains why we refer to L_S as the countable Lawvere theory for global state: taking $C = Set$, the induced monad is the monad for global state or side-effects proposed by Moggi [10,12].

Proposition 2.5 *The left adjoint of the forgetful functor*

$$U : Mod(L_S, C) \longrightarrow C$$

sends an object X of C to the object $(S \otimes X)^S$ together with the maps

$$a_{T_S(X)} : ((S \otimes X)^S)^1 \longrightarrow ((S \otimes X)^S)^{\mathrm{Loc} \times V}$$

determined, modulo two isomorphisms, by composition with the co-operation

$$a_S : (\mathrm{Loc} \times V) \times S \longrightarrow 1 \times S$$

obtained by transposition from $g_{:=}$ that, given a pair $((loc, v), \sigma)$, "updates" $\sigma \in S = V^{\mathrm{Loc}}$ by replacing its value at loc by v and

$$d_{T_S(X)} : ((S \otimes X)^S)^V \longrightarrow ((S \otimes X)^S)^{\mathrm{Loc}}$$

determined, modulo two isomorphisms, by composition with the co-operation

$$d_S : \mathrm{Loc} \times S \longrightarrow V \times S$$

obtained by transposition from $g_!$ *that, given a pair* (loc, σ)*, "lookups"* loc *in* $\sigma \in S = V^{\mathrm{Loc}}$ *to determine its value.*

More generally, for any countable Lawvere theory L, one can consider the tensor product $L \otimes L_S$ [3,4]. The monad induced by $L \otimes L_S$ on Set is $T_L(S \times -)^S$. And the L_S-model structure of Proposition 2.5 extends from $(S \times X)^S$ to $T_L(S \times X)^S$ to give the L_S-model structure on the free $(L \otimes L_S)$-model $T_L(S \times X)^S$ on any set X: the key point is that the model structure is determined entirely in terms of the exponent S.

Restating a result in [9], if C is a category with S-fold powers and copowers, and with a regular epi-mono factorisation system, then the functor $(-)^S : C \longrightarrow C$ is monadic, making C equivalent to $Mod(L_S, C)$ coherently with respect to $(-)^S : C \longrightarrow C$. So every model of L_S in C has the form Y^S, with operations determined by composition, as in Proposition 2.5; and maps of models $Y^S \to Z^S$ are given precisely by maps of the form f^S. This result evidently applies to the category Set.

Moreover, since we know from [3] that $Mod(L \otimes L_S, Set)$ is equivalent to $Mod(L_S, Mod(L, Set))$ then, as $Mod(L, Set)$ satisfies the conditions of the theorem, see [1], $Mod(L \otimes L_S, Set))$ is canonically equivalent to $Mod(L, Set)$. So every model of $Mod(L \otimes L_S, Set)$ has the form A^S, with L_S-structure given as before and with L-structure given pointwise, and maps of models $A^S \to B^S$ are given precisely by maps of the form f^S. Unfortunately, because of the regularity requirement of [9], these results do not immediately enrich in the usual way.

Example 2.6 Let L_r denote the countable Lawvere theory for read-only state. It is freely generated by a map $r : S_r \longrightarrow 1$, where S_r is a non-empty countable set, subject to the commutativity of two diagrams, which, expressed as infinitary equations, become:

(i) $r((x)_s) = x$

(ii) $r(\,(r(\,(x_{ss'})_s\,))_{s'}\,) = r((x_{ss})_s)$

The induced monad on Set is $(-)^{S_r}$.

The free L_r-model on a set X is X^{S_r}, with the L_r-model structure on X^{S_r} given by precomposition with the diagonal

$$(X^{S_r})^{S_r} \cong X^{S_r \times S_r} \xrightarrow{X^{\delta}} X^{S_r}$$

So again, the model structure is determined entirely in terms of the exponent S_r. And again, that extends to tensor products $L \otimes L_r$, the free $(L \otimes L_r)$-model on a set X being given by $(T_L X)^{S_r}$. We note finally that in the case where $S_r = V^{\mathrm{Loc}}$, the theory L_r can, alternately, be presented by an operation $d : V \to \mathrm{Loc}$ subject to equations (ii), (v) and (viii) above.

The final example of a countable Lawvere theory of primary importance to us here is that of a monoid action; it has several computational applications.

Example 2.7 Given a monoid M, the countable Lawvere theory L_M induces the monad $M \times -$ on Set; the theory is generated by M unary operations f_m, respecting the monoid structure of M, i.e., $f_e = id$ where e is the unit of M, and $f_m f_{m'} = f_{mm'}$ where the multiplication of M is denoted by juxtaposition. The category of models of L_M in Set is the category of left M-sets. For an arbitrary countable Lawvere theory L, the tensor product of L with L_M generates the monad $T_L(M \times -)$ on Set [4].

One use of this theory is for resources, e.g., timed processes [4,8]; there the monoid is typically the positive reals, or the natural numbers, with addition. Another is write-only memory, where for example, one takes the theory generated by an operation $a : 1 \to \mathrm{Loc} \times V$ and equations (iii) and (vi) above; in this case the monoid has carrier $\sum_{L \subseteq \mathrm{Loc}} V^L$.

3 Comodels of Countable Lawvere Theories

In this section, we briefly recall from [16] the notion of a comodel of an arbitrary countable Lawvere theory, focusing upon the example L_S of global state in Section 2. The abstract results of this section again enrich routinely to categories that are locally countably presentable as cartesian closed categories.

Definition 3.1 A *comodel* of a countable Lawvere theory L in a category C with countable coproducts is a countable coproduct preserving functor from L^{op} to C.

Comodels of L in a category C with countable coproducts form a category $Comod(L, C)$, whose arrows are given by all natural transformations. So, almost by definition, for any category C with countable coproducts, we have the following:

$$Comod(L, C) \cong Mod(L, C^{op})^{op}$$

It follows from Theorem 2.2 that for any countable Lawvere theory L and any category C for which C^{op} is locally presentable, the forgetful functor $U : Comod(L, C) \longrightarrow C$ has a right adjoint, exhibiting $Comod(L, C)$ as comonadic over C. But Set^{op} is not locally presentable, so this fact is of no help in regard to our leading example of a base category.

Nevertheless, the following is true, as shown in [16].

Theorem 3.2 *For any countable Lawvere theory L, the forgetful functor*

$$U : Comod(L, Set) \longrightarrow Set$$

has a right adjoint, exhibiting $Comod(L, Set)$ as comonadic over Set.

The central fact yielding the proof is the cartesian closedness of Set, specifically the fact that the tensor of an object a of L, i.e., a countable set, with

X has the universal property of a product. It follows that one can extend the result to categories such as *Poset*, ω*Cpo*, and *Cat*, and that it generalises to enrichment in any category that is locally countably presentable as a cartesian closed category, such as *Poset*, ω*Cpo*, or *Cat*.

Dually to models, comodels have pointwise sums, so that

$$(C + C')(a) = C(a) + C'(a)$$

and similarly for morphisms. For any set X and comodel C, we write $X \times C$ for the X-fold sum of C, so that $(X \times C)(a) = X \times C(a)$ and similarly for morphisms.

The main theorem of [16] asserted that the category of comodels of the countable Lawvere theory L_S is given by a category of arrays. We summarise the situation briefly.

Definition 3.3 Given a non-empty finite set Loc of locations and a non-empty countable set V of values, a (Loc, V)-*array* consists of a set A together with functions

$$sel : A \times \mathrm{Loc} \longrightarrow V$$

and

$$upd : A \times \mathrm{Loc} \times V \longrightarrow A$$

subject to four axioms written in equational form as follows: for l and l' in Loc, for v and v' in V, and for a in A

(i) $sel(upd(a, l, v), l) = v$

(ii) $upd(a, l, sel(a, l)) = a$

(iii) $upd(upd(a, l, v), l, v') = upd(a, l, v')$

(iv) $upd(upd(a, l, v), l', v') = upd(upd(a, l', v'), l, v)$ where $l \neq l'$

There is an evident notion of a map of arrays, yielding a category (Loc, V)-*Array*. The relationship between the definition of an array and the countable Lawvere theory L_S is not entirely trivial. Nevertheless, the central technical result of [16] was as follows.

Theorem 3.4 *The forgetful functor from* (Loc, V)*-Array to Set is comonadic over Set, with comonad given by* $(-)^{V^{\mathrm{Loc}}} \times V^{\mathrm{Loc}}$. *Moreover,* (Loc, V)*-Array is equivalent to Set, with the forgetful functor from* (Loc, V)*-Array to Set given by* $- \times V^{\mathrm{Loc}}$.

The category Set^{op} has countable products and countable coproducts, with products given by the coproducts of *Set* and with coproducts given by the products of *Set*. So, combining Theorem 2.4 with Theorem 3.4, we reached the desired conclusion as follows.

Corollary 3.5 *Let* L_S *be the countable Lawvere theory for global state. Then* $Comod(L_S, Set)$ *is equivalent to* (Loc, V)*-Array.*

It follows from Theorem 3.4 and Corollary 3.5 that the final L_S-comodel is given by the set S of states with its canonical co-operations

$$a_S : (\mathrm{Loc} \times V) \times S \longrightarrow S \qquad d_S : \mathrm{Loc} \times S \longrightarrow V \times S$$

obtained by transposition from the generic effects $g_{:=}$ and $g_!$, and further explained in Proposition 2.5; we may denote this comodel by S, relying on context to disambiguate. More generally, it follows that every comodel is of the form $X \times S$ and that the maps of comodels from $X \times S$ to $Y \times S$ are of the form $f \times S$.

Theorem 3.4 and Corollary 3.5 enrich routinely, assuming one uses the evident definition of enriched (Loc, V)-array.

Our next two classes of examples are read-only state, extending Example 2.6, and a monoid action, extending Example 2.7.

Example 3.6 A priori, to give a comodel for L_r is to give a set X together with a function $X \longrightarrow S_r \times X$ subject to two axioms. But the unit axiom simply asserts that the projection to X yields the identity, and the composition axiom is trivial. So a comodel is just a function $X \longrightarrow S_r$, i.e., an object of the slice category Set/S_r. The maps work similarly. The final comodel is therefore given by the set S_r together with the identity map. In the enriched setting, one generalises from Set/S_r to V/S_r: an object of a slice V-category C/X is defined to be an arrow with codomain X in the V_0-category C.

Example 3.7 For a monoid M, the Lawvere theory L_M is generated by unary operations subject to axioms that dualise: the duality is given by swapping left with right in M. So, to give a comodel of L_M is equivalent to giving a model of L_M but with the order of multiplication reversed, making $Comod(L_M, Set)$ the category of right M-sets. The final comodel is 1, but a more interesting comodel C_M is given by the set M with action determined by the multiplication of M together with a twist. This example also enriches routinely.

Finally, we note that if a theory contains a constant or a commutative binary operation, then its only comodel in Set is trivial, with empty carrier. We therefore do not expect coalgebra to play any direct rôle in such computational effects as exceptions or ordinary or probabilistic nondeterminism.

4 The Tensor of a Comodel with a Model

In this section, for any countable Lawvere theory L, we describe a tensor $C \otimes M$ of an arbitrary comodel C with an arbitrary model M and calculate it in several cases. At the heart of our category theoretic analysis is the fundamental theorem that asserts that the Yoneda embedding expresses the presheaf category $[D^{op}, Set]$ as the free cocompletion of any small category D:

Theorem 4.1 *Let D be a small category. Then for any cocomplete locally small category E, composition with the Yoneda embedding*

$$\mathrm{Y} : D \longrightarrow [D^{op}, Set]$$

induces an equivalence of categories

$$Cocomp([D^{op}, Set], E) \cong [D, E]$$

where, for any cocomplete locally small category E', the category $Cocomp(E', E)$ is the category of colimit preserving functors from E' to E and all natural transformations between them.

A proof of this appears in the enriched setting in Kelly's book [6], in which it plays a central rôle. The inverse equivalence sends a functor $H : D \longrightarrow E$ to its left Kan extension $Lan_{\mathrm{Y}} H$, which can be described in more elementary terms as follows:

$$(Lan_Y H)(F) = \int^{d \epsilon D} Hd \times Fd \tag{5}$$

where, for any set X, we write $Hd \times X$ for the X-fold coproduct of Hd. This is a coend, so is given by factoring the sum $\Sigma_{d \epsilon D}(Hd \times Fd)$ by dinaturality: H is covariant in D and F is contravariant in D, so any map $f : d' \longrightarrow d$ generates two functions

$$(Hf \times Fd), (Hd' \times Ff) : Hd' \times Fd \longrightarrow Hd \times Fd'$$

and one factors the sum $\Sigma_d(Hd \times Fd)$ by the equivalence relation $\sim$ generated by all such pairs of functions, yielding

$$\int^{d \epsilon D} Hd \times Fd = (\Sigma_{d \epsilon D}(Hd \times Fd))/\sim \tag{6}$$

So Theorem 4.1 asserts that every colimit preserving functor from $[D^{op}, Set]$ to E is isomorphic to one that sends F in $[D^{op}, Set]$ to $(\Sigma_d(Fd \times Hd))/\sim$ for some functor $H : D \longrightarrow E$, uniquely up to coherent isomorphism.

The theorem says a little more than that in that the fully faithfulness part of being an equivalence says that natural transformations are respected by the constructs too. One can make a slightly stronger statement. All colimit preserving functors from $[D^{op}, Set]$ to E have right adjoints, and those adjoints can be described as follows: for any functor $H : D \longrightarrow E$, the functor sending an object X of E to $E(H-, X) : D^{op} \longrightarrow Set$ is the right adjoint to $Lan_Y H$.

There are numerous refinements of Theorem 4.1. A refinement in the direction we need appears in [7] and tells us the following.

Theorem 4.2 *Let D be a small category with countable coproducts. Then for any cocomplete locally small category E, composition with the Yoneda embedding*

$$\mathrm{Y} : D \longrightarrow CP(D^{op}, Set)$$

induces an equivalence of categories

$$Cocomp(CP(D^{op}, Set), E) \cong CC(D, E)$$

where $CP(D^{op}, Set)$ denotes the full subcategory of $[D^{op}, Set]$ determined by the countable product preserving functors from D^{op} to Set, and $CC(D, E)$ denotes the category of countable coproduct preserving functors from D to E.

There are a number of subtleties implicit in the statement of Theorem 4.2. First, for any small category D with countable coproducts, the Yoneda embedding

$$\mathrm{Y} : D \longrightarrow [D^{op}, Set]$$

factors through $CP(D^{op}, Set)$: that part is easy. Second, the restricted variant of the Yoneda embedding, i.e., the Yoneda embedding regarded as having codomain $CP(D^{op}, Set)$, preserves countable coproducts: that follows from the Yoneda lemma. Third, the category $CP(D^{op}, Set)$ is cocomplete: that is a substantial result, the colimits not being given pointwise in general.

Given the statement, the proof of the theorem is not difficult. Moreover, the formula for the inverse equivalence is identical to that for Theorem 4.1, i.e., a countable coproduct preserving functor $H : D \longrightarrow E$ corresponds to the colimit preserving functor from $CP(D^{op}, Set)$ to E given by restricting $Lan_{\mathrm{Y}}H$ from $[D^{op}, Set]$ to $CP(D^{op}, Set)$, thus sending F in $CP(D^{op}, Set)$ to the coend (5). The right adjoint also restricts, with the same formula as in the classical case.

Observe that a countable Lawvere theory L is a small category with countable products, the category $Mod(L, Set)$ is exactly $CP(L, Set)$, and the category $CC(L^{op}, E)$ is the category of comodels of L in E. Thus we have:

Corollary 4.3 *Let L be a countable Lawvere theory. Then for any cocomplete locally small category E, the Yoneda embedding*

$$\mathrm{Y} : L^{op} \longrightarrow Mod(L, Set)$$

induces an equivalence of categories

$$Cocomp(Mod(L, Set), E) \cong Comod(L, E)$$

The above analysis gives a formula for the inverse equivalence, i.e., (5), more explicitly (6), as well as a right adjoint $E \longrightarrow Mod(L, Set)$.

We now specialise to the case $E = Set$: given a comodel C and a model M of L we write $C \otimes M$ for (the set) $Lan_{\mathrm{Y}}C(M)$. Thus, by (5), we have the following formula:

$$C \otimes M = \int^{a \epsilon L} Ca \times Ma \tag{7}$$

It follows from its definition that the construction $C \otimes M$ is bifunctorial.

The objects of L are exactly the natural numbers together with $\aleph_0$. So each object a is a countable coproduct in $\aleph_1$, equivalently a countable product in L, of a copies of 1. Since C preserves countable coproducts and M preserves countable products, we have:

$$Ca \times Ma \cong a \times C1 \times M1^a$$

Next, by the Yoneda lemma, the coend

$$\int^{a \epsilon \aleph_1^{op}} a \times CJ1 \times MJ1^a$$

is $C1 \times M1$, where J is the canonical functor from $\aleph_1^{op}$ into L, and so $C \otimes M$ is given by further coequalising $C1 \times M1$ with respect to arbitrary maps in L yielding the universal map

$$\otimes_{C,M} : C1 \times M1 \longrightarrow C \otimes M$$

discussed in the introduction; see too the discussion just before Theorem 6.2.

Given a comodel $C : L^{op} \longrightarrow Set$ of L in Set, as we know, the right adjoint of $C \otimes -$ sends a set Y to the composite

(8) $$L \xrightarrow{C} Set^{op} \xrightarrow{Y^-} Set$$

necessarily a model of L in Set. This 'exponential' construction of a model from a comodel was exemplified in Section 2 for state and read-only state.

The next theorem gives an explicit form for the tensor and its universal map in a particular case. We write ltr for the *left transpose* map

$$Set(X \times Y, Z) \rightarrow Set(Y, Z^X)$$

Note that the construction of the map uses the above exponential construction of a model.

Theorem 4.4 *For any countable Lawvere theory L, comodel C of L and set X, the tensor $C \otimes T_L X$ of C with the free model $T_L X$ of L on X is given by the product $C1 \times X$ and $\otimes_{C,T_L(X)} : C1 \times T_L(X) \rightarrow C1 \times X$ is given by the formula $ltr^{-1}((ltr(id_{M1\times X}))^{\dagger})$.*

Proof. For any set Y, considering Y^{C1} as an L-model as above, the set of maps $C1 \times T_L(X) \rightarrow Y$ coequalising $C1 \times T_L(X)$ with respect to arbitrary maps in L is in bijective correspondence with $Mod(L, Set)(T_L(X), Y^{C1})$, via transposition. But the latter set is in evident bijective correspondence with $Set(X, Y^{C1})$ and so, by inverse transposition, with $Set(C1 \times X, Y)$. □

In the case of global state L_S, with $C = S$, one then has that the tensor is $S \times X$ and the universal map is application, just as in the informal analysis of the introduction. Indeed, as the reader may check, all the the other examples of tensor we calculate also accord with the usual practice in operational semantics.

Our next example is read-only state. By Example 2.6, the free $(L \otimes L_r)$-model of $L \otimes L_r$ on a set X is given by $(T_L X)^{S_r}$, which is also the free L_r-model on the set $T_L X$. Thus Theorem 4.4 applies, making $S_r \otimes (T_L X)^{S_r} = S_r \times T_L X$ and the universal map $(s, f) \mapsto (s, fs)$. One usually combines exceptions with other effects by considering the sum of theories $L' + L_E$. This corresponds to the monad $T_{L'}(- + E)$ on Set. So the theorem also applies if we add exceptions as the monad induced by $(L \otimes L_r) + L_E$ is given by $(T_L(X + -))^{S_r}$.

Lemma 4.5 *Suppose we are given a comodel C and a model M of a countable Lawvere theory L. Suppose too that we have the tensor $C \otimes M$, together with its universal map $\otimes_{C,M} : C1 \times M1 \longrightarrow C \otimes M$. Then the tensor of $X \times C$ with M, together with its universal map, is given by:*

$$(X \times C)1 \times M1 \;\cong\; X \times (C1 \times M1) \xrightarrow{X \times \otimes_{C,M}} X \times (C \otimes M)$$

5 Further Calculations of the Tensor

In this section, we extend tensors for Theorem 4.4 for global state and monoid actions. As in the case of read-only state, we are interested in combinations with other theories by taking the tensor of theories, and with exceptions by taking the sum of theories [4]. Our calculations again enrich without fuss.

First consider L_S. Recall that this has final comodel S and that, for any set Y, there is a model Y^S.

Theorem 5.1 *For any set Y, $S \otimes Y^S$ is Y, and the universal map $S \times Y^S \to Y$ is given by application.*

Proof. It suffices to prove that evaluation generates an isomorphism of sets

$$\int^{a \epsilon L_S} (Ca \times Y^{Ca}) \longrightarrow Y$$

which follows from a mild strengthening of a form of the Yoneda embedding to account for the *density* of the functor C, equivalently of the full subcategory of Set given by those sets of the form $a \times S$ [6]. Alternatively, one can prove the result directly: for any element y of Y, consider constant functions at y and use the fullness of the functor C. □

Applying Lemma 4.5, for any sets X and Y the tensor $(X \times S) \otimes Y^S$, together with its universal map, is given by:

$$(X \times S) \times Y^S \;\cong\; X \times (S \times Y^S) \xrightarrow{X \times \mathrm{ev}} X \times Y$$

Since $(X \times S)$ and Y^S are the general forms of comodels and models of L_S, this gives a complete picture of the tensor for the theory of state; however, note that, as remarked above this may not enrich.

The tensor product $L \otimes L_S$, with L a countable Lawvere theory, generates the monad $T_L(S \times -)^S$ [3,4]. As remarked in Section 2, the L_S-model structure on $T_L(S \times X)^S$ smoothly generalises that on $(S \times X)^S$: it is determined by the final comodel structure of the exponent S. So Theorem 5.1 yields:

Corollary 5.2 *For any countable Lawvere theory L and any set X, if C is the final comodel of L_S and M is the free $(L \otimes L_S)$-model on X, the tensor $C \otimes M$ is the set $T_L(S \times X)$ and the universal map is application.*

We actually know a little more, that if M is any $(L \otimes L_S)$-model of the form A^S then $(X \times C) \otimes A^S$ is (the set) $X \times A$. Further, this gives a complete picture of the tensor of comodels of state with models of state tensored with another theory; however this last statement may not enrich.

We can squeeze a little more value out of Theorem 5.1 to obtain the formula $T_L(S \times (X + E))$ for the tensor of the final comodel of L_S with the free $((L \otimes L_S) + L_E)$-model on any set X.

For monoid actions L_M, our primary interest lies not in the trivial final comodel, but rather in the monoid M treated as the comodel C_M.

Theorem 5.3 *For any countable Lawvere theory L and any set X, if M' is the free $(L \otimes L_M)$-model on X, the tensor $C_M \otimes M'$ is the set $T_L(M \times X)$ and the universal map $\otimes : M \times T_L(M \times X) \to T_L(M \times X)$ is $T_L(\mu_{T_M})\circ st_L$.*

Proof. Since $\aleph_1$ is included in L_M, there is a canonical function

$$\int^{a \epsilon \aleph_1^{op}} (M \times a) \times (T_L(M \times X))^{M \times a} \longrightarrow \int^{a \epsilon L_M} Ma \times M'a$$

The Yoneda lemma applied to $\aleph_1$ implies the following, for any set Y:

$$Y \cong \int^{a \epsilon \aleph_1} (a \times Y^a)$$

Given X, putting $Y = (T_L(M \times X))^M$, using cartesian closedness of Set and the formula (7) for tensor, the above two displays yield a function

$$M \times T_L(M \times X) \longrightarrow C_M \otimes M'$$

exhibiting $C_M \otimes M'$ as a quotient of $M \times T_L(M \times X)$.

The quotient is generated by the identification of the pairs $(m, \eta(e, x))$ and $(e, \eta(m, x))$ for any m in M, where e is the unit of M. That routinely yields the result, the coprojections being given by the canonical strength of T_L together with the multiplication of M. $\square$

Theorem 5.3 yields the formula $T_L(M \times (X + E))$ for the tensor of C_M and the free $((L \otimes L_M) + L_E)$-model on a set X.

6 Combining Comodels

In previous sections, we have considered comodels of three main theories: L_S, L_r and L_M. But one may have more than one of these acting at once, for instance employing triples (s, t, M) consisting of a state s, a time t and a term M. So in this section, we consider a tensorial combination of comodels and its interaction with the tensor with models.

Observe that, since Set is cartesian closed, for any pair of comodels C of L and C' of L' in Set, the functor

$$L^{op} \times L'^{op} \xrightarrow{C \times C'} Set \times Set \xrightarrow{\times} Set$$

preserves countable coproducts in each argument separately, i.e., for every $a \epsilon L$, the functor $Ca \times C'(-)$ preserves countable coproducts, and dually. By the universal property of the tensor product of countable Lawvere theories, the composite thus yields a comodel of $L \otimes L'$ in Set, which we shall denote by $C \circ C'$: so $(C \circ C')1 = C1 \times C'1$, with the co-action on $C1 \times C'1$ given by multiplying the $C1$ co-action by $C'1$ and the dual.

Theorem 4.4 immediately yields a formula as follows:

Corollary 6.1 *For any countable Lawvere theories L and L' with comodels C and C' respectively in Set, if M is the free $(L \otimes L')$-model on a set X, the tensor $(C \circ C') \otimes M$ is given by $C1 \times C'1 \times X$.*

We next consider models that need not be free $(L \otimes L')$-models on a set X. Given countable Lawvere theories L and L', denote the coprojections from L and L' into $L \otimes L'$ by J and J' respectively. So, for any model M of $L \otimes L'$, it follows that MJ is a model of L and MJ' is a model of L'.

Theorem 6.2 *For any countable Lawvere theories L and L', comodels C of L and C' of L', and model M of $L \otimes L'$, the tensor $(C \circ C') \otimes M$ is the pushout in Set given as follows:*

$$\begin{tikzcd}
C1 \times C'1 \times M1 \arrow[r, "s \times M1"] \arrow[d, "C1 \times \otimes_{C',MJ'}"'] & C'1 \times C1 \times M1 \arrow[r, "C'1 \times \otimes_{C,MJ}"] & C'1 \times (C \otimes MJ) \arrow[d] \\
C1 \times (C' \otimes MJ') \arrow[rr] & & (C \circ C') \otimes M
\end{tikzcd}$$

Proof. The following formulae are consequences of formula (7) together with cartesian closedness of Set:

$$\int^{a \epsilon \aleph_1^{op}} (a \times C1 \times C'1 \times (M1)^a) = C1 \times C'1 \times M1$$

$$\int^{a \epsilon L} (a \times C1 \times C'1 \times (M1)^a) = C'1 \times (C \otimes MJ)$$

$$\int^{a \epsilon L'} (a \times C1 \times C'1 \times (M1)^a) = C1 \times (C' \otimes MJ')$$

$$\int^{a \epsilon L \otimes L'} (a \times C1 \times C'1 \times (M1)^a) = (C1 \circ C') \otimes M$$

Every map in $L \otimes L'$ is a composite of a map in L with a map in L'. The result follows by an elementary colimit calculation. □

Evidently, one can consider more than two countable Lawvere theories and their comodels. But our analysis here has only involved routine manipulation of colimits and the cartesian closedness of Set. So we leave it to the reader to formulate associativity results and the like.

Example 6.3 Let C and C' be the final comodels S and S_r for state and read-only state respectively, and let M_1 be the free $(L_S \otimes L_r \otimes L)$-model $T_L(S \times X)^{S \times S_r}$ on a set X. Then, putting $Y = T_L(S \times X)$ and suppressing the canonical twist map, the tensor $(C \circ C') \otimes M_1$ is the pushout

$$\begin{tikzcd} S \times S_r \times Y^{S \times S_r} \arrow[r, "S_r \times ev_S"] \arrow[d, "S \times (\pi_{S_r}, ev_{S_r})"'] & S_r \times Y^{S_r} \arrow[d, "\rho_0"] \\ S \times S_r \times Y^S \arrow[r, "\rho_1"'] & P \end{tikzcd}$$

We need therefore only show that the commutative diagram obtained by replacing P by $S_r \times Y$ satisfies the universal property of a pushout. So let s_0 be a chosen element of S. Given (s', h) and (s', h') in $S_r \times Y^{S_r}$ such that $h(s') = h'(s')$, consider the two elements of $S \times S_r \times (Y^{S_r})^S$ determined by s_0, s', and the constants at h and h' respectively. Suppressing a canonical isomorphism, the function $S \times (\pi_{S_r}, ev_{S_r})$ identifies those two elements. So ρ_0 must identify (s', h) with (s', h'). So the pushout is indeed given by $S_r \times Y$.

Thus $(C \circ C') \otimes M_1$ is $S_r \times T_L(S \times X)$ and $(s, s', f) \mapsto (s', f(s, s'))$ is the universal map.

It is routine to extend Example 6.3 to incorporate an M-action, yielding the following formula:

$$(S \circ S_r \circ C_M) \otimes T_L(S \times M \times X)^{S \times S_r} = S_r \times T_L(S \times M \times X)$$

for global state S and read-only state S_r, together with an evident formula for the universal map. The calculation in Example 6.3 also goes through for ωCpo (taking advantage of the fact that ρ_0 is a retraction) as does the extension (for the corresponding reason).

7 Concluding Remarks

Having available the theory of the tensor of a comodel with a model, the next thing to do is to prove an adequacy result generalising that in [11]. One general possibility is to consider a theory of the form:

$$L_E + ((L_S \otimes L_r \otimes L_M) \otimes L)$$

and the previous section shows how to calculate the relevant tensor. Such a result would cover many cases of interest.

One that it would not is that of resumptions which, in the case of Set, corresponds to the theory:

$$L_d + (L_S \otimes L_N)$$

where L_d is the theory of a unary operation d with no equations and L_N is the theory of a semilattice, to model nondeterministic choice; see [4] for more details. However, in this case the operational semantics itself becomes more complicated involving not a single evaluation to produce a value, together with a final state, but a series of them to account for interruption points where parallel programs may execute.

One might well seek a general operational semantics for some pleasant abstract structure, say two theories, one included in the other and a comodel for the included one. But the example of resumptions hints that carrying out such a program would not be a trivial matter.

In another direction, it may be that the calculations of tensors could be improved. They were carried out above on rather a case to case basis, whether individually or in combination. Perhaps there is a more uniform approach?

Finally it would be interesting to incorporate other comodels. One source is provided by automata. For example, consider a (finite) automaton (S, Σ, δ, F) where $\delta : \Sigma \times S \to S$ and $F \subset S$ and the initial state is omitted. Such an automaton can be regarded as a coalgebra on the set of states S with δ a co-operation $\Sigma \times S \to S$ and F a co-operation $S \to 2 \times S$. Any coalgebra C with countable sorts naturally determines a countable Lawvere theory L_C which can be presented by the equations true in the coalgebra in a suitable sense. Having L_C available, one may then seek to apply the above ideas.

A further question is how to incorporate 'automata with effects'. For example a probabilistic finite automaton can be considered as a coalgebra in the Kleisli category of the distributions with finite support monad [11]. This raises interesting questions regarding the relevant theories and tensor calculations.

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A Comodel-Model Biactions

One can give a more general setting for our construction of the tensor $C \otimes M$ of a comodel C with a model M that allows a form of iteration of the process. The tensor $C \otimes M$ can be seen as an instance of composition in a naturally existing bicategory: for any set A with both an L-model structure and an L'-comodel structure on it, subject to natural coherence axioms, and any set B with an L'-model structure and an L''-comodel structure subject to similar coherence conditions, one can build a composite $A \otimes B$ that factors out the L'-structures while inheriting the L-structure of A and the L''-structure of B.

We do not yet have any application of this compositional generalisation of tensor, but as it is mathematically relevant and substantial, we include analysis of it in this section of the paper. We should perhaps mention that, for the enriched setting, one needs a notion of enriched bicategory here that is routine to formulate, but goes beyond the currently standard literature.

Definition A.1 Let CC denote the 2-category for which

- 0-cells are small categories with countable coproducts

- 1-cells are functors that preserve countable coproducts
- 2-cells are all natural transformations

with the evident composition.

The forgetful 2-functor $U : CC \longrightarrow Cat$ has a left biadjoint F, meaning F is essentially a left adjoint but adjusted to deal with non-identity isomorphisms and with 2-cells [2].

Passing over size concerns, which can be treated by recourse to Section 2 of [6] for example, it follows from Theorem 4.2 that the construction that sends a small category with countable coproducts D to $CP(D^{op}, Set)$ extends canonically to a pseudo-monad T_{coc} on CC. Consider the bicategory $Kl(T_{coc})$. In particular, consider two 0-cells of it. One of them is $F1$, the free category with countable coproducts on 1. So $F1$ is equivalent to Nat, but we shall not use that fact here. The other is L^{op} for any countable Lawvere theory L seen as a small category with countable products as in Corollary 4.3.

Straightforward calculations show the following:

Proposition A.2 *In the bicategory $Kl(T_{coc})$*

(i) *to give a 1-cell from $F1$ to L^{op} is equivalent to giving a model of L in Set*

(ii) *to give a 1-cell from L^{op} to $F1$ is equivalent to giving a comodel of L in Set*

(iii) *to give a 1-cell in $Kl(T_{coc})$ from $F1$ to $F1$ is equivalent to giving a set.*

The discussion after Theorem 4.2 may be rephrased as the statement that composition in $Kl(T_{coc})$ is calculated pointwise, i.e., the inclusions

$$CP(D^{op}, Set) \longrightarrow [D^{op}, Set]$$

generate a 2-functor from $Kl(T_{coc})$ to $Prof$, the 2-category of small categories, profunctors, and natural transformations, where composition is calculated as follows: given $H : D \times D'^{op} \longrightarrow Set$ and $K : D' \times D''^{op} \longrightarrow Set$, the composite $H \otimes K : D \times D''^{op} \longrightarrow Set$ sends (x, y) to the coend

$$\int^{z \epsilon D'} H(x, z) \times K(z, y)$$

Thus, applying Proposition A.2, given a model $M : L \longrightarrow Set$ and a comodel $C : L^{op} \longrightarrow Set$ of a countable Lawvere theory L, the composite in $Kl(T_{coc})$ yields the set $\int^{a \epsilon L} Ca \times Ma$, recovering the tensor formula (7).

To give a 1-cell in $Prof$ from D to D' is equivalent to giving a functor $H : D \times D'^{op} \longrightarrow Set$, which is equivalent to giving a 1-cell in $Prof$ from D'^{op} to D^{op}. This fact has proved to be of considerable value in the abstract theory of categories, supporting Street's study of two-sided fibrations [17]. Unfortunately, the 2-category $Kl(T_{coc})$ does not allow such symmetry. For to give a countable coproduct preserving functor from D to $CP(D'^{op}, Set)$ is not equivalent to giving a functor from $D \times D'^{op}$ to Set that preserves countable coproducts in its first argument and countable products in its second. Moreover, we cannot see any malleable formulation of the 1-cells of $Kl(T_{coc})$ in such terms. So we see no malleable way in which to treat the 1-cells of $Kl(T_{coc})$ in terms of two-sided fibrations [17] satisfying natural conditions related to preservation of products and coproducts.