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FAST COMMUNICATION

A STABILITY RESULT FOR SOLITARY WAVES IN NONLINEAR DISPERSIVE EQUATIONS*

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Abstract. The stability of solitary traveling waves in a general class of conservative nonlinear dispersive equations is discussed. A necessary condition for the exchange of stability of traveling waves is presented; an unstable eigenmode may bifurcate from the neutral translational mode only at relative extrema of the wave energy. This paper extends a result from Hamiltonian systems, and from a few integrable partial differential equations, to a broader class of conservative differential equations, with particular application to gravity-capillary surface waves.

Key words. stability, solitary wave, gravity-capillary wave.

AMS subject classifications. 76B45, 76B25, 76B15

1. Introduction

This paper investigates the linear stability of nonlinear traveling waves in dispersive evolution equations. It is well known that the derivative of a solution to a translation-invariant equation is a neutrally stable eigenfunction of the linearized equation. The question addressed here is under what circumstances may the eigenvalue for this eigenfunction, referred to here as the translational mode, bifurcate from zero. This type of analysis originated in Hamiltonian systems, where Saffman proved that the linear stability of traveling waves in a family of waves may only change at local extrema of the speed-energy curve [1]. More recently, similar results have been shown for the the Nonlinear Schrödinger (NLS), Korteweg-de Vries (KdV), and Kadomtsev-Petviashvilli (KP) equations [2, 3]. Here, we extend this result to classes of one-way and bidirectional conservative, nonlinear, dispersive equations. The result applies to models for gravity-capillary waves, where the dynamics and stability of solitary waves is of current interest [4, 5]. In the gravity-capillary case, the bifurcation studied in this paper has been observed numerically [6]. The understanding of this observed bifurcation was the motivation for this paper.

2. Model equations and solitary waves

Consider the first order in time evolution equation of the form

$$\eta_t - i\Omega\eta + \frac{1}{2}Q^*(PQ\eta)_x^2 = 0. \tag{2.1}$$

Here, x is the preferred direction of propogation. In this variable, Ω has a real, odd Fourier symbol; P has a real, even Fourier symbol, and it is self-adjoint. The operator Q has an arbitrary Fourier symbol; Q^* is the adjoint operator to Q whose Fourier multiplier is the complex conjugate of $\hat{Q}$. Equation (2.1) conserves at least $M = \int \eta$ and $E = \int \eta P\eta$, and is translation-invariant. For example, when $Q = P = 1$ and

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$\hat{\Omega} = k^3$ (where k is the Fourier dual variable to x) equation (2.1) is the KdV equation. When $\hat{\Omega} = sgn(k)k^2$, it is the (nonlocal) Benjamin-Ono equation. Equations in more than one spatial dimension are also amenable to the analysis we present. For example, with $Q = P = 1$ and $\hat{\Omega} = k^3 \mp l^2/k$ (where l is the Fourier dual variable to y), we get the KP-I and KP-II equations. For deep water gravity-capillary waves one may derive models using the full dispersion relation $\hat{\Omega} = \mathrm{sign}(k)\sqrt{|k|(1+k^2)}$; examples of such models are derived in [6].

We also consider the second order in time model

$$\phi_{tt} + \Omega^2\phi + Q^*(PQ\phi_t)_x^2 = 0, \tag{2.2}$$

which is also translation-invariant, and conserves at least $M_2 = \int \phi_t$ and $E_2 = \int \phi_t P\phi_t + \phi P\Omega^2\phi$. The operators Q, P and Ω satisfy the same conditions as in equation (2.1). The operator Ω^2 can thus correspond to any even, real, positive Fourier multiplier. In the case of deep water gravity-capillary waves, $\hat{\Omega}^2 = |k|(1+|k|^2)$.

Equ. (2.2) can be factored symmetrically into left and right moving waves.

$$2u = \phi_t + i\Omega\phi, \qquad 2v = \phi_t - i\Omega\phi,$$

so

$$\phi_t = u + v, \qquad i\Omega\phi = u - v.$$

In terms of these new variables, equation (2.2) becomes

$$u_t - i\Omega u + \frac{1}{2}Q^*(PQ(u+v))_x^2 = 0, \tag{2.3a}$$

$$v_t + i\Omega v + \frac{1}{2}Q^*(PQ(u+v))_x^2 = 0. \tag{2.3b}$$

If, at initial times, the waves are primarily propagating in one direction (for example, $v(x,t)$ is small, then, depending on Q, P and Ω, these equations may decouple for long times as in the gravity-capillary case [6]. Neglecting v, equation (2.3a) reduces to equation (2.1).

The goal of this work is to investigate the stability of *finite-amplitude*, fully localized solitary waves (also called lump solutions in higher dimensions). We are particularly interested in wave-packet type waves: those that bifurcate from linear wavepackets at a finite wave-number. The equations that support such waves are usually non-local and non-integrable. Wave-packet solitary waves have been found, for example, as solutions to the free-surface Euler equations in deep water with surface tension, and in a few model equations (for two dimensional examples, see [6, 3, 5, 7]). For $\hat{\Omega} = \mathrm{sign}\sqrt{|k|(1+k^2)}$, both (2.1) and (2.2) have such solitary wave solutions for many choices of P and Q. Examples of these waves for one choice of P and Q that conserves momentum in the fluid (see [6]) are plotted in Fig. 2.1. A diagnostic tool for the existence of such waves at *small amplitude* in a particular evolution equation is the existence of a *focusing* NLS equation governing wave-packets near the wavenumber where the group and phase speed are equal [6, 8, 7].

3. Stability of traveling waves

The stability of solitary waves has been of long standing interest [9, 10]. In previous work, the authors presented three simple one-dimensional one-way models for infinite depth gravity-capillary waves, together with numerical computations of their stability and dynamics [6]. It was found that the stability of these waves can

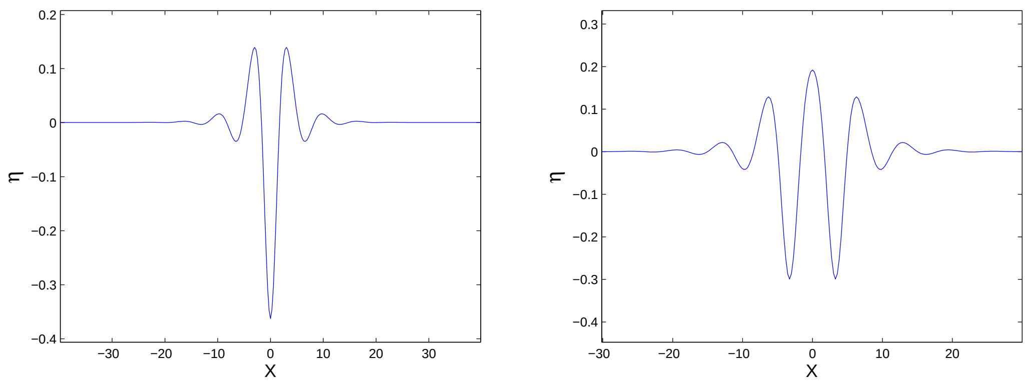

FIG. 2.1. *A depression (left) and an elevation (right) solitary wave solution to equation (2.1), with* $\Omega = sign(k)\sqrt{|k|(1+k^2)}$, $P = |\Omega|$, *and* $Q = |\Omega|^{-1}$.

change as the amplitude increases. The purpose of this work is to find a criterion for the stability change. The authors have proposed models for two dimensional waves [11], which support lump solutions, and the present analysis applies to those also.

To begin, consider equation (2.1) in the frame of an exact solution moving with speed c,

$$\eta_t - c\eta_x - i\Omega\eta + \frac{1}{2}Q^*(PQ\eta)_x^2 = 0, \tag{3.1}$$

and suppose that $\bar{\eta}(x)$ is a steady solution to (3.1), that is, a traveling solution to (2.1). Consider perturbations of the form $\eta = \bar{\eta} + \epsilon\rho(x)e^{\lambda t}$. The linearization of (3.1) is

$$\lambda\rho - c\rho_x - i\Omega\rho + Q^*(PQ\bar{\eta}PQ\rho)_x = 0. \tag{3.2}$$

Equ. (3.2) is of the form $\mathcal{L}\rho = \lambda\rho$. This equation has the well known solution $\lambda = 0$, $\rho^{(0)} = \bar{\eta}_x$, due to the translation-invariance of equation (3.1). The goal is now to understand how this solution behaves in a neighborhood of $\lambda = 0$. In particular, we want to determine when eigenmodes with real nonzero eigenvalues bifurcate from the translational mode. First, we assume that $|\lambda| \ll 1$ and express ρ as a power series in λ about $\rho^{(0)}$.

$$\rho = \rho^{(0)} + \lambda\rho^{(1)} + \lambda^2\rho^{(2)} + \ldots \tag{3.3}$$

Collecting powers of λ transforms the linear eigenvalue problem into a series of linear problems with the eigenvalue removed. The leading order equation is the original linear problem for the zero eigenvalue, which has solution $\rho^{(0)} = \bar{\eta}_x$. At $O(\lambda)$, the leading order solution occurs as an inhomogeneous term in the equation for $\rho^{(1)}$:

$$\mathcal{L}\rho^{(1)} = \bar{\eta}_x. \tag{3.4}$$

This equation has, as a particular solution, $\rho^{(1)} = -\frac{\partial\bar{\eta}}{\partial c}$, which can be seen by differentiating equation (3.1) with respect to c. At order $O(\lambda^2)$ the first correction, $\rho^{(1)}$, forces the equation for $\rho^{(2)}$:

$$\mathcal{L}\rho^{(2)} = -\bar{\eta}_c. \tag{3.5}$$

Using the Fredholm alternative, a solvability condition for equation (3.5) is that $\bar{\eta}_c$ is orthogonal to solutions of the adjoint equation

$$cv_x + i\Omega v - Q^* P(PQ\bar{\eta}Qv_x) = 0. \tag{3.6}$$

One solution to (3.6) is $v = P\bar{\eta}$. The corresponding solvability condition is

$$\int \bar{\eta}_c P\bar{\eta}dx = \frac{1}{2}\frac{d}{dc}\int \eta P\eta = \frac{1}{2}\frac{dE}{dc} = 0.$$

We conclude that a necessary condition for the translational eigenmode to become unstable is that the energy is at a local extremum, as a function of the wave speed.

The same technique can be used to derive stability conditions for second order in time equations — by factoring into a system of first order in time equations, and proceeding as before. We consider the second order in time model (2.2) which conserves $M_2 = \int \phi_t$ and $E_2 = \int \phi_t P\phi_t + \phi P\Omega\phi$. Rewriting (2.2):

$$\phi_t = v \tag{3.7a}$$
$$v_t = -\Omega^2\phi + Q^*(PQv)^2_x. \tag{3.7b}$$

Changing to a traveling frame, traveling wave solutions $(\bar{\phi}, \bar{v})$ to (3.7) are now steady solutions to (3.8).

$$\phi_t - c\phi_x = v \tag{3.8a}$$
$$v_t - cv_x = -\Omega^2\phi + Q^*(PQv)^2_x \tag{3.8b}$$

Linearizing about these solutions, writing $(\phi, v) = (\bar{\phi}, \bar{v}) + \delta\left(\Phi(x)e^{(\lambda t)}, V(x)e^{(\lambda t)}\right)$ yields

$$\lambda\Phi - c\Phi_x = V \tag{3.9a}$$
$$\lambda V - cV_x = -\Omega^2\Phi + 2Q^*(PQ\bar{v}PQV)_x. \tag{3.9b}$$

As before, equation (3.9) is of the form

$$\mathcal{A}\begin{pmatrix}\Phi\\ V\end{pmatrix} = \lambda\begin{pmatrix}\Phi\\ V\end{pmatrix}.$$

We write both Φ and V as a series in λ, about the translational solution $(\bar{\phi}_x, \bar{v}_x)$

$$\Phi = \bar{\phi}_x + \lambda\Phi_1 + \lambda^2\Phi_2 + ...$$
$$V = \bar{v}_x + \lambda V_1 + \lambda^2 V_2 + ...$$

At $O(\lambda)$ we get a forced linear system.

$$\mathcal{A}\begin{pmatrix}\Phi_1\\ V_1\end{pmatrix} = \begin{pmatrix}\bar{\phi}_x\\ \bar{v}_x\end{pmatrix} \tag{3.10}$$

Equ. (3.10) has solution $(\Phi_1, V_1) = -(\frac{\partial\bar{\phi}}{\partial c}, \frac{\partial\bar{v}}{\partial c})$. At $O(\lambda^2)$, the solutions (Φ_1, V_1) force the equation for (Φ_2, V_2)

$$\mathcal{A}\begin{pmatrix}\Phi_2\\ V_2\end{pmatrix} = -\begin{pmatrix}\bar{\phi}_c\\ \bar{v}_c\end{pmatrix}. \tag{3.11}$$

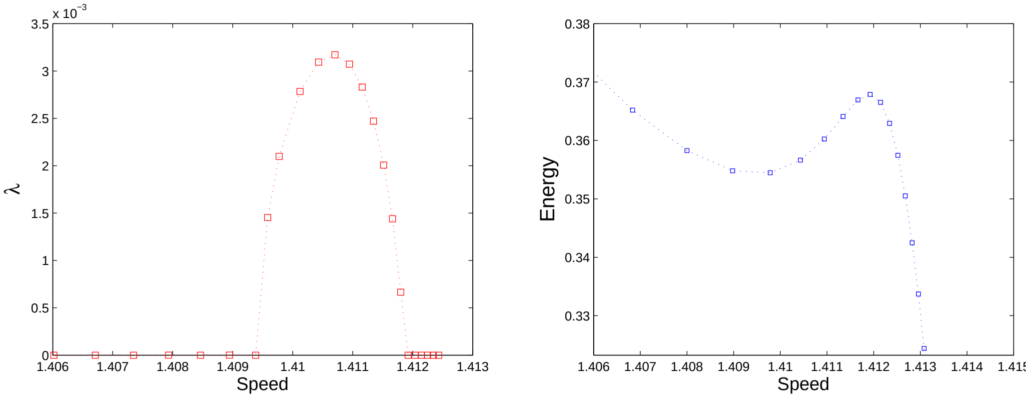

FIG. 3.1. ***Left:*** *The real positive eigenvalue for depression waves in equation (2.1), with* $\Omega = sign(k)\sqrt{|k|(1+k^2)}$, $P = (1-\partial_x^2)$ *and* $Q = (1-\partial_x^2)^{-1}$, *plotted as function of the wave speed.* ***Right:*** *The energy of depression waves in this equation plotted as a function of the wave speed. The solution bifurcates from a linear wavepacket at* $c = \sqrt{2}$.

For equation (3.11) to have solutions, a necessary condition is that the vector $(\frac{\partial\bar{\phi}}{\partial c}, \frac{\partial\bar{v}}{\partial c})$ is orthogonal to solutions to the adjoint equation

$$\mathcal{A}^* \begin{pmatrix} \psi \\ u \end{pmatrix} = \begin{pmatrix} c\partial_x & \Omega^2 \\ -1 & c\partial_x - 2Q^*P(PQ\bar{v}Q\partial_x) \end{pmatrix} \begin{pmatrix} \psi \\ u \end{pmatrix} = 0. \tag{3.12}$$

One solution to (3.12) is $(\psi, u) = (P\Omega^2\bar{\phi}, P\bar{v})$. The corresponding solvability condition is

$$\int \frac{\partial\bar{v}}{\partial c}P\bar{v} + \frac{\partial\bar{\phi}}{\partial c}P\Omega^2\bar{\phi} = \frac{1}{2}\frac{d}{dc}\left(\int \phi_t P\phi_t + \phi P\Omega^2\phi\right) = \frac{1}{2}\frac{dE_2}{dc} = 0.$$

This stability condition allows one to understand the linear stability of traveling waves via the graph of the speed versus the relevant conserved quantity, as in Fig. 3.1 (right). The stability of traveling solitary waves of elevation and depression for two equations of the form of (2.1) was investigated numerically in [6]. The linear spectra were computed, and the instabilities were observed via numerical time evolution. For equation (2.1) with Ω, P, Q as defined in Fig. 2.1 the waves of elevation were unstable at all amplitudes, and those of depression were stable. However, in an example with $\hat{\Omega} = \mathrm{sign}(k)\sqrt{|k|(1+k^2)}$, $\hat{P} = (1+k^2)$, and $\hat{Q} = (1+k^2)^{-1}$, which corresponds to a model of gravity-capillary waves that conserves exactly the quadratic fluid energy $E = ||u||_{H^1}^2 = \int uPu = \int u^2 + u_x^2$, the stability is quite different. Depression waves have a linear spectrum which is purely imaginary for both small and large amplitude waves, but waves of intermediate amplitudes have a real positive eigenvalue which bifurcates from zero at the local extrema of the energy. The unstable eigenvalue for this example equation is plotted in Fig. 3.1, together with the speed-energy curve. The long time evolution of this instability is a time-periodic traveling solution, orbiting the stable state at the energy level of the unstable wave [6].

Note that neither derivation of the stability criterion relies on the whether ϕ or η are functions of one or more space dimensions. In this case we assume that $\hat{\Omega}$ is odd in the vector $\mathbf{k}$ (for real η and ϕ). In two-dimensions, if $\hat{\Omega}$ is odd in k, then it must be even in l. If it does not vanish at $k = 0$ then extra conditions must be imposed on the initial data's mean in x, as in KP. There also may be additional conditions on P and Q that we do not consider.

Examples of two dimensional equations for which our result applies are the KP-I equation,

$$u_t - \frac{1}{6}u_{xxx} + \frac{1}{2}\partial_x^{-1}u_{yy} + \frac{3}{2}uu_x = 0, \tag{3.13}$$

which is integrable and for which lumps are always stable, and the fifth-order KP-equation,

$$u_t + \frac{1}{6}u_{xxx} + \frac{1}{90}u_{xxxxx} + \frac{1}{2}\partial_x^{-1}u_{yy} + \frac{3}{2}uu_x = 0, \tag{3.14}$$

for which lumps seem to have more complicated stability characteristics [3]. Both apply to a rather unphysical situation of gravity-capillary waves in shallow water based on the surface tension length scale: the former applies to a regime of $B > 1/3$ and the latter for $B \approx 1/3$, where B is the Bond number [12]. These equations govern long waves in x with even weaker variation in y, and their linear parts are obtained by expanding the dispersion relation about $\mathbf{k} = (k_x, k_y) = 0$. For gravity-capillary waves on infinite depth, one can also derive a KP-like equation by expanding about $\mathbf{k}= (k_x, k_y) = (1, 0)$, and imposing some global conditions on the dispersion relation. The result is

$$u_t + \frac{\sqrt{2}}{4}\mathcal{H}u - \frac{\sqrt{2}}{4}\mathcal{H}u_{xx} + \epsilon\left(\frac{\sqrt{2}}{2}\mathcal{H}u_{yy} - \frac{3}{4\sqrt{2}}uu_x\right) = 0. \tag{3.15}$$

The operator $\mathcal{H}$ is the Hilbert transform in x, whose Fourier symbol is $\hat{\mathcal{H}} = -i\text{sign}(k)$. The forgoing analysis predicts the change of stability of lump solutions to this equation. An example lump solitary wave solution to equation (3.15) and its speed-energy plot are in Fig. 3.2. The authors present the derivation of equation (3.15), as well as stability and dynamics of solitary waves in [13].

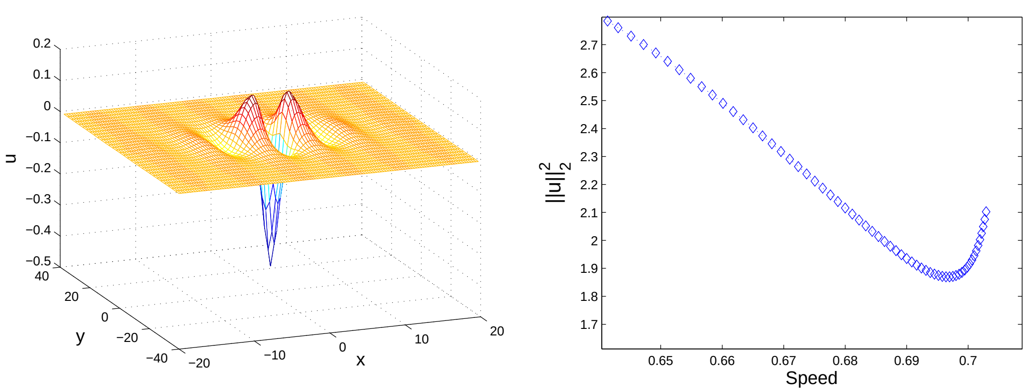

Fig. 3.2. ***Left:*** *An example of a lump depression wave solution to equation (3.15).* ***Right:*** *The speed-energy plot for depression lump solitary wave solutions to equation (3.15). The nonlinear solution bifurcates from a linear wavepacket at* $c = \sqrt{2}/2$.

4. Conclusion

A necessary condition on the bifurcation of unstable eigenvalues for the translational eigenfunction of traveling waves in one-way and bidirectional classes of dispersive nonlinear equations was derived. Bifurcations can occur only at local extrema in the graph of the speed versus the quadratic conserved quantity. Solitary waves are computed for an example equation, and the linear stability changes exactly at the local extremum in the speed-energy plot.

This note focuses on equations with quadratic nonlinearities. One-way and bidirectional equations can be written with nonlinearities of higher degree. The result can be trivially extended to equations with higher power nonlinearities, for example

$$\eta_t + i\Omega\eta + \sum_n a_n Q^*(PQ\eta)_x^n = 0 \tag{4.1}$$

with a_n arbitrary constants.

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