Algebraic properties of the Lambert $W$ Function from a result of Rosenlicht and of Liouville

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Abstract

It is shown that the Lambert $W$ function cannot be expressed in terms of the elementary, Liouvillian, functions. The proof is based on a theorem due to Rosenlicht. A related function, the Wright $\omega$ function is similarly shown to be not Liouvillian.

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The Lambert $W$ function [5, 9] is a multi-valued function defined as the solution of

$$W(x)e^{W(x)} = x,$$  \hspace{1cm} (1)

one of the simplest possible non-algebraic equations. The Wright $\omega$ function [4] also satisfies a simple transcendental equation (away from its discontinuities):

$$\omega(x) + \ln \omega(x) = x.$$  \hspace{1cm} (2)

Both of these functions are implicitly elementary, in the sense discussed by Risch in [7]. One can ask whether there are explicit formulations of those functions

\hspace{1cm} *This paper is dedicated to the memory of Manuel Bronstein (1963–2005).
in terms of known functions, or whether they are genuinely new functions. A common class of “well-known” functions are the Liouvillian functions.

**Definition 1** Let \((k,')\) be a differential field of characteristic 0. A differential extension \((K,')\) of \(k\) is called Liouvillian over \(k\) if there are \(\theta_1,\ldots,\theta_n \in K\) such that \(K = C(x,\theta_1,\ldots,\theta_n)\) and for all \(i\), at least one of the following holds:

1. \(\theta_i\) is algebraic over \(k(\theta_1,\ldots,\theta_{i-1})\);
2. \(\theta'_i = \eta\) for some \(\eta \in k(\theta_1,\ldots,\theta_{i-1})\);
3. \(\theta'_i/\theta_i = \eta\) for some \(\eta \in k(\theta_1,\ldots,\theta_{i-1})\).

We say that \(f(x)\) is a Liouvillian function if it lies in some Liouvillian extension of \((C(x),d/dx)\) for some constant field \(C\).

It turns out that the possible closed-form expressions for solutions of equations of the form \((1–2)\) were already studied by Liouville [6], who was certainly able to prove already that \(W(x)\) is not a Liouvillian function. In any event, this result was known to Rosenlicht, who published in [8] a proposition that can be applied to prove easily that \(W(x)\) and \(\omega(x)\) (or many functions defined by similar transcendental equations) are not Liouvillian. Yet, questions about whether \(W(x)\) is elementary or Liouvillian appear in the literature [3], possibly because Rosenlicht’s paper is not as well-read as it deserves to be, so we illustrate in this note how Rosenlicht’s theorem can prove that neither \(W(x)\) nor \(\omega(x)\) are Liouvillian.

We start by recalling Rosenlicht’s result.

**Proposition 1** [8, Proposition, p.21] Let \(k\) be a differential field of characteristic zero and let \(y_1,\ldots,y_n,z_1,\ldots,z_n\) be elements of a Liouvillian extension of \(k\) having the same subfield of constants as \(k\). Suppose that

\[
\frac{y'_i}{y_i} = z'_i, \quad i = 1,\ldots,n,\]

and that \(k(y_1,\ldots,y_n,z_1,\ldots,z_n)\) is algebraic over each of its subfields \(k(y_1,\ldots,y_n)\) and \(k(z_1,\ldots,z_n)\). Then, \(y_1,\ldots,y_n,z_1,\ldots,z_n\) are all algebraic over \(k\).

An immediate consequence of the case \(n = 1\) of that proposition is that if \(W(x)\) and \(\omega(x)\) are Liouvillian functions, then they must be algebraic functions: suppose that \(W\) belongs to a Liouvillian extension \(K\) of \(C(x)\). Take \(k = C(x)\) where \(C\) is the constant subfield of \(K\), then \(K\) is Liouvillian over \(k\) and both fields have the same subfield of constants. Taking logarithmic derivatives on both sides of \((1)\) yields

\[
W'/W + W'/W' = 1/x, \quad (3)
\]

whence \(y'/y = W'\) where \(y = x/W \in K\). Since \(k(y,W) = k(y) = k(W)\), Rosenlicht’s theorem implies that \(W\) is algebraic over \(k = C(x)\). The proof is similar for \(\omega(x)\): differentiating both sides of \((2)\) yields \(\omega' + \omega'/\omega = 1\), whence
\( \omega'/\omega = z' \) where \( z = x - \omega \). Since \( k(\omega, z) = k(\omega) = k(z) \), Rosenlicht’s theorem implies that \( \omega \) is algebraic over \( k = C(x) \).

There are obvious analytic arguments why \( W(x) \) and \( \omega(x) \) cannot be algebraic functions, so they cannot be Liouvillian functions: if \( W(x) \) has a pole of finite order, then \( e^{W(x)} \), and therefore \( W(x)e^{W(x)} \), have an essential singularity, so \( W(x)e^{W(x)} \) cannot equal \( x \). Similarly if \( \omega(x) \) has a zero, then \( \ln \omega(x) \), and therefore \( \omega(x) + \ln \omega(x) \), have a logarithmic singularity, so \( \omega(x) + \ln \omega(x) \) cannot equal \( x \). Since algebraic functions with either no pole or no zero must be constants, and \( W(x) \) and \( \omega(x) \) cannot be constant, they cannot be algebraic.

The above argument can be cast in algebraic terms. Since Rosenlicht proved this result algebraically, we outline the algebraic proof that \( W(x) \) and \( \omega(x) \) cannot be algebraic functions. Note that (3) implies that \( y = W(x) \) is a solution of the differential equation

\[
y' = x(1 + y).
\] (4)

We first recall some notations and results from [2]: we say that a field \( E \) is an algebraic function field of one variable over a subfield \( F \subset E \) if

- \( E \) is of transcendence degree 1 over \( F \),
- for any \( t \in E \) transcendental over \( F \), \([E : F(t)]\) is finite.

By an \( F \)-place of \( E \), we then mean the maximal ideal of a valuation ring of \( E \) containing \( F \). For such a place \( p \), we write \( \nu_p : E^* \to \mathbb{Z} \) for its order function. It has in particular the following properties:

- \( \nu_p(c) = 0 \) for any \( c \in \overline{F} \cap \mathbb{Z}^* \).
- \( \nu_p(ab) = \nu_p(a) + \nu_p(b) \) and \( \nu_p(a + b) \geq \min(\nu_p(a), \nu_p(b)) \) for any \( a, b \in E^* \).
- \( \nu_p(a + b) = \min(\nu_p(a), \nu_p(b)) \) for any \( a, b \in E^* \) such that \( \nu_p(a) \neq \nu_p(b) \).
- For any \( a \in E^* \), if \( \nu_p(a) \geq 0 \) at all the \( F \)-places of \( E \), then \( a \) is algebraic over \( F \).

Let now \( t \in E \) be transcendental over \( F \) and \( p \) be any \( F \)-place of \( E \). We write \( r_p(t) \in \mathbb{Z}^* \) for the ramification index of \( p \) over \( F(t) \). In addition, we call the place \( p \) infinite (w.r.t. \( t \)) if \( t^{-1} \in p \), finite (w.r.t. \( t \)) otherwise. A finite place \( p \) contains a unique monic irreducible \( P \in F[t] \), called the center of \( p \) (w.r.t. \( t \)).

**Proposition 2** Let \( (F', x') \) be a differential field containing an element \( x \) such that \( x' = 1 \). If \( F \) has transcendence degree 1 over its constant subfield, then the only solution \( y \in F \) of (4) is \( y = 0 \).

**Proof.** Let \( C \) be the constant subfield of \( F \) and suppose that \( F \) has transcendence degree 1 over \( C \). Since \( x' = 1 \), \( x \) is transcendental over \( C \), so \( F \) is algebraic over \( C(x) \). Let \( y \in F \) be a nonzero solution of (4) and \( E = \mathbb{C}(x, y) \), which is an algebraic function field of one variable over \( C \). Let \( p \) be any \( \mathbb{C} \)-place of \( E \). Applying \( \nu_p \) on both sides of (4), we get

\[
\nu_p(x) + \nu_p(y') + \nu_p(1 + y) = \nu_p(y). \tag{5}
\]
Suppose that $\nu_p(y) < 0$. Then, $\nu_p(1 + y) = \min(0, \nu_p(y)) = \nu_p(y)$ and (5) becomes

$$\nu_p(x) + \nu_p(y') = 0.$$  \hspace{1cm} (6)

If $p$ is finite w.r.t. $x$, then $\nu_p(x) \geq r_x(p)$. But Lemma 1.7 of [1] implies that $\nu_p(y') = \nu_p(y) - r_x(p) < -r_x(p)$, in contradiction with (6). If $p$ is infinite, then $\nu_p(x) = -r_x(p)$. But Lemma 1.8 of [1] implies that $\nu_p(y') \leq \nu_p(y) + r_x(p) < r_x(p)$, in contradiction with (6). Therefore $\nu_p(y) \geq 0$ at all the $\mathbb{C}$-places of $E$, which implies that $y \in \mathbb{C}$, hence that $y' = 0$, and (4) becomes $0 = y$. \hfill \Box

Since the only algebraic solution of (4) is 0, which is not a solution of (1), $W(x)$ cannot be algebraic, hence it cannot be a Liouvillian function.

The proof that $\omega(x)$ is not an algebraic function is similar, since $y = \omega(x)$ is a solution of the differential equation $y'(1 + y) = y$. The equalities (5) and (6) become respectively $\nu_p(y') + \nu_p(1 + y) = \nu_p(y)$ and $\nu_p(y') = 0$, and the proof of Proposition 2 remains valid.

References


