Mixing Closures for Conservation Laws in Stratified Flows

By Tivon Jacobson, Paul A. Milewski, and Esteban G. Tabak

A closure for shocks involving the mixing of the fluids in two-layer stratified flows is proposed. The closure maximizes the rate of mixing, treating the dynamical hydraulic equations and entropy conditions as constraints. This closure may also be viewed as yielding an upper bound on the mixing rate by internal shocks. It is shown that the maximal mixing rate is accomplished by a shock moving at the fastest allowable speed against the upstream flow. Depending on whether the active constraint limiting this speed is the Lax entropy condition or the positive dissipation of energy, we distinguish precisely between internal hydraulic jumps and bores. Maximizing entrainment is shown to be equivalent to maximizing a suitable entropy associated to mixing. By using the latter, one can describe the flow globally by an optimization procedure, without treating the shocks separately. A general mathematical framework is formulated that can be applied whenever an insufficient number of conservation laws is supplemented by a maximization principle.

1. Introduction

The vertical stratification of the atmosphere and ocean is the result of the competing effects of complex mechanisms that enhance stratification, such as differential heating, and mixing processes, which uniformize the density. Quantifying these mixing processes is challenging because of the scale

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disparity between the large-scale fluid motion and the diffusion processes that ultimately lead to mixing. The energy transfer between these scales occurs through turbulent motion, created either by fluid instabilities or by breaking internal waves. Ideally, one would like to be able to include the effects of mixing on the large scale dynamics without resolving the detailed dynamics of turbulence. The possibility that we explore here is to replace this detailed dynamics by an optimization procedure, that maximizes an appropriate measure of mixing while satisfying constraints imposed by the large-scale dynamics.

In this paper, we develop these ideas in the context of mixing at the interface between two fluid masses of different densities due to the overturning and breaking of internal waves. This scenario may play a role in a number of significant applications to geophysical flows, including the entrainment of deeper, interior fluid into the ocean's surface mixed layer [1], the entrainment of ambient water into dense overflows downstream of oceanic sills [2], and the entrainment of stratospheric air through the tropopause into the troposphere [3]. The Earth's rotation plays a significant role in most geophysical applications. The aim of this paper, however, is to isolate and model the physics of entrainment, and therefore we study idealized scenarios without rotational effects.

A conclusion of geophysical relevance that our model provides is a precise mathematical distinction between two kinds of internal breaking waves, hydraulic jumps and bores, whose different structure was first pointed out in [4]. With this distinction, it becomes clear that breaking waves at the base of the ocean mixed layer can be characterized as internal *bores*, while those occurring downstream of sills are better described as internal *hydraulic jumps*.

We use a one-and-a-half layer shallow water model: the situation that we consider is that of a shallow layer of fluid below a slightly lighter, deep, ambient fluid at rest (see Figure 1). (The reverse situation of a thin layer of light fluid above a deep heavy fluid can be mapped to the one we consider.) Because we consider mostly shock-type solutions, we assume that the fluid is bounded by horizontal walls, although the addition of topography is straightforward. We consider the fluids to be *miscible*, and this paper is an effort to propose deductive closures that quantify the entrainment and mixing due to shocks.

Even though mixing in stratified fluids is often modeled in terms of shear instability, and quantified through the Richardson number Ri, shocks may play a more important role [5–7]. We have shown [8] that the evolution equations for two-layer shallow water are nonlinearly stable up to wave breaking, if the initial conditions have $Ri < 1$. Hence in this case breaking waves provide the only strong mechanism for mixing.

In Section 2, we derive the set of partial differential equations for the dynamics of the layer in terms of u, a uniform or average horizontal velocity, h, the layer depth and b the buoyancy, proportional to the density of the fluid. There are two conservation laws: total mass and momentum. These are valid even at possible discontinuities in the solutions (shocks). If one

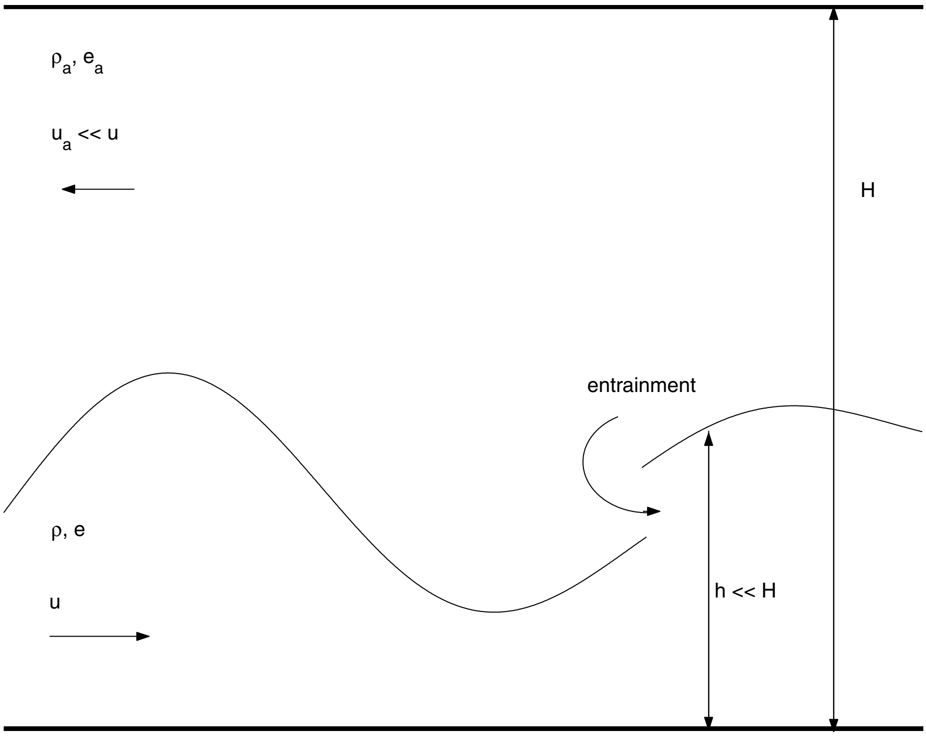

Figure 1. Two-layer flow with entraining hydraulic jump.

postulates that the fluids do not mix (constant b), then the equations become the well known shallow water equations. When the waves break, the two conservation laws define the shock speed and predict the amount of wave energy dissipated at the shock and transformed to turbulent energy. Here we consider the case in which the fluids do mix. In this case, it is useful to introduce a third conservation principle: that of *total* energy, the sum of energy in the shallow water dynamics and the small scale turbulent energy created by shocks. We now have three conservation laws for four dynamical variables. We therefore need to add another postulate: that there is no small scale energy created in smooth parts of the flow. The result of this is a closed hyperbolic system for smooth parts of the flow, derived in Section 3, and the requirement for a closure that sets the amount of mixing at shocks. In Section 4, we propose such a closure: by arguing that shocks moving at their maximal allowable speed (constrained by the Lax condition) will maximize entrainment.

In Section 5, we explore the consequences and range of validity of this closure, and in Section 6 we prove that it does indeed maximize entrainment. The closure is shown to apply to situations where the flow is supercritical upstream of the shock, as is the case in standing hydraulic jumps. For these, we quantify precisely the amount of mixing at the shock and show that the shocks are physical, in the sense that they also dissipate large scale energy and create turbulence.

The optimality of the closure is proved only locally in Section 6, that is, among shock speeds close to the maximal allowable one. Section 7 proves that the closure is in fact globally optimal, and does it in a quite general framework, that allows to extend this result to other incomplete systems of conservation laws. In Section 8, we introduce a mixing *entropy*, and show that maximizing entropy generation is equivalent to maximizing entrainment.

Our closure yields physically relevant shocks and predictions of entrainment and generation of turbulence in a range of regimes which we call internal hydraulic jumps: regimes in which energy is dissipated in the lower layer. When this ceases to be the case, the Lax condition on allowed shock speeds is no longer an active constraint, and is replaced by the constraint of nonnegative energy dissipation. Hence the corresponding entrainment maximizing shocks have *no* dissipated energy in the lower layer, although they still entrain and mix. These are the internal hydraulic bores, first distinguished from hydraulic jumps in [4]. Physical examples of bores include gravity currents [9, 10] and, we conjecture, the breaking waves at the base of the ocean mixed layer. (Notice that, in regular shallow water, bores and hydraulic jumps are mathematically equivalent, because they can be mapped into each other through a Galilean transformation. This is not the case for one-and-a-half layer flows, where the ambient layer fixes a preferred frame of reference.)

The results above all decompose the flow into smooth parts, where a close set of partial differential equations apply, and discontinuities, where the standard jump conditions based on conservation laws are completed by the maximization requirement. It would be desirable, however (especially for numerical purposes), to be able to treat the whole flow in a unified fashion, proposing a global maximization principle. This is what we propose in Section 9, where we include the smooth parts of the flow into the optimization procedure. This sheds new light into the physical distinction between hydraulic jumps and bores.

The work presented here has a dual character. On the one hand, it proposes a closure for entraining breaking waves, under a somewhat idealized set of assumptions. One interpretation of the closure and its consequences is that we find an *upper bound* for the amount of mixing at a shock allowed by the dynamical equations of motion. On the other hand, this work is presented as an application of a more general proposal for quantifying mixing in large-scale models, through an optimization procedure constrained by the coarse-grained equations of motion. Consequently, Sections 1–6 are a self-contained discussion of our mixing closure at jumps, while Sections 7–9 frame this discussion in a more abstract and generalizable setting.

2. Conservation principles

Consider the configuration depicted in Figure 1, of a two-layer flow, in a channel with rigid top and bottom lids, separated by a distance H. Because the top layer will be assumed to be much deeper than the bottom one, and thought of as an ambient, we will use the subindex a to identify the corresponding variables, such as the velocity and density, while no subindices will be used for the variables representing quantities associated with the bottom, active layer. The ambient density ρ_a is constant and uniform, but ρ may vary, due to the entrainment process. The main modeling assumptions that we make, some of which are already implicit in Figure 1, are the following:

- The flow is mainly horizontal, and consists of two layers of vertically homogeneous density and velocity. That is, even though entrainment of parcels of ambient fluid into the lower layer will be considered, these are assumed to almost instantly mix throughout the lower layer, driven by turbulent motion.
- No detrainment (parcels of fluid from the lower layer entering the ambient) occurs.
- The pressure for the large-scale motion is hydrostatic. Within shocks, the pressure is clearly nonhydrostatic. In a description in terms of conservation laws, however, these nonhydrostatic effects are encompassed by the jump conditions at shocks that our closure will specify.
- The density difference between the two layers is small.

In the limit of a very deep ambient, this two-layer system may be modeled quite simply, in terms of dynamical variables associated with the bottom layer alone. Then, conservation of buoyancy (volume and mass), momentum and energy read

$$(bh)_t + (bhu)_x = 0 \tag{1}$$

$$(hu)_t + \left(hu^2 + \frac{bh^2}{2}\right)_x = 0 \tag{2}$$

$$\left(\frac{hu^2}{2} + \frac{bh^2}{2} + he\right)_t + \left(\frac{hu^3}{2} + bh^2u + heu\right)_x = 0. \tag{3}$$

Here, h is the depth of the active layer, u its mean velocity, $b = g\frac{\rho - \rho_a}{\rho_a}$ is a normalized density (the letter b stands for "buoyancy", though the latter is better represented by $-b$; "reduced gravity" is another commonly used name for b), and e is the kinetic energy associated to turbulent motion.

We sketch here briefly a derivation of the system (Equations (1)–(3)). A key element of this derivation is that, because entrainment brings along exchanges of volume, mass, momentum and energy between the two layers, one cannot apply the associated conservation principles to each layer individually. Instead, these principles have to be posed globally, for the two layers at once. In a fixed domain confined by rigid lids, conservation of volume adopts the simple form

$$(hu + (H - h)u_a)_x = 0, \tag{4}$$

yielding

$$hu + (H - h)u_a = Q(t), \tag{5}$$

where $Q(t)$, the volume flux through the system, is so far undetermined function of time. Conservation of mass, on the other hand, reads

$$(\rho h + \rho_a(H - h))_t + (\rho hu + \rho_a(H - h)u_a)_x = 0, \tag{6}$$

which combined with (4) yields the buoyancy Equation (1). To write down the momentum equation, we need the vertical integral of the [hydrostatic] pressure, given by

$$\int_0^H p\,dz = \frac{1}{2}g\rho_a H^2 + PH + \frac{1}{2}b\rho_a h^2,$$

where P denotes the pressure at the top lid. The vertical integral of the momentum flux is

$$\int_0^H \rho u^2\,dz = \rho_a(H - h){u_a}^2 + \rho hu^2.$$

The resulting momentum equation equates the time evolution of the local momentum density to horizontal variations in the pressure and the momentum flux:

$$(\rho hu + \rho_a(H - h)u_a)_t + \left(\rho_a(H - h){u_a}^2 + \rho hu^2 + \frac{1}{2}b\rho_a h^2\right)_x + HP_x = 0.$$

To eliminate the extra unknown P, we argue, as in [4], that, in the absence of detrainment, the ambient flow is comparatively smooth from some streamline up, even above internal hydraulic jumps. Then we have, on the uppermost streamline, that

$$\rho_a(u_{at} + u_a u_{ax}) + P_x = 0, \tag{7}$$

which reduces the momentum equation to

$$(\rho hu - \rho_a hu_a)_t + \left(\rho hu^2 - \rho_a h{u_a}^2 + \frac{1}{2}b\rho_a h^2 + \rho_a H\frac{{u_a}^2}{2}\right)_x = 0. \tag{8}$$

Finally we consider the limit of a deep ambient, $H \gg h$. For the global volume flux $Q(t)$ in (5) to remain bounded, we need to have $u_a = O(1/H)$

which, together with the Boussinesq approximation, reduces the momentum Equation (8) to its simpler form in (2).

Notice that the simplification brought about by this assumption of a very deep ambient carrying a bounded volume flux comes at a cost: By making the ambient fluid nearly motionless, we are effectively selecting a distinguished reference frame. Hence the Equations (1) and (2) are not Galilean invariant, unlike their shallow-water counterpart with constant b.

The derivation of the energy Equation (3) follows steps very similar to the momentum equation, and will be skipped here for brevity (a full derivation may be found in [11] or, for the particular case of standing internal hydraulic jumps, in [12]).

Because Equations (1), (2), and (3) are valid even at mixing internal shocks, they yield the jump conditions

$$-c[bh] + [bhu] = 0 \tag{9}$$

$$-c[hu] + \left[hu^2 + \frac{bh^2}{2}\right] = 0 \tag{10}$$

$$-c\left[\frac{hu^2}{2} + \frac{bh^2}{2} + he\right] + \left[\frac{hu^3}{2} + bh^2u + heu\right] = 0, \tag{11}$$

where c is the shock's speed, and the brackets indicate the jump of the enclosed expression across it. We denote with subscripts $-$ and $+$ the states up and downstream of the shock respectively. In Figure 1, upstream is to the left and downstream to the right of the shock, because the height h is higher on the right, and the fluid crosses the jump from left to right. Notice that we have four active variables: b, h, u and e, and only three equations. Hence the need for an additional physical principle to close the system.

3. Smooth evolution

In this article, we focus on the intense and localized mixing taking place at internal breaking waves. Hence we simplify matters by assuming that no mixing takes place away from shocks. Other types of mixing that could be considered separately can arise from the Kelvin–Helmholz instability or from other external sources of turbulent energy.

There are various ways to formulate this assumption: in terms of the buoyancy b, the turbulent energy e, or the depth h. In terms of the buoyancy, no mixing in smooth parts of the flow implies that buoyancy is advected,

$$b_t + ub_x = 0, \tag{12}$$

or equivalently, from (1), volume conservation:

$$h_t + (hu)_x = 0. \tag{13}$$

On the other hand, in smooth parts of the flow, the advection of turbulent energy is given by

$$e_t + ue_x = \frac{h}{2}\left(1 - F^2 + \frac{2e}{hb}\right)(b_t + ub_x),$$

which follows from (1), (2), and (3), with $F = \frac{u}{\sqrt{bh}}$ the Froude number. Hence a statement equivalent to (12) is that the internal energy density is advected,

$$e_t + ue_x = 0. \tag{14}$$

This in turn, together with (1) and (12) yields an alternative formulation, i.e., conservation of turbulent energy:

$$(he)_t + (heu)_x = 0, \tag{15}$$

Even though Equations (13) and (15) are written in conservation form, they are not meant to be valid at discontinuities of the flow, across which turbulence is created, volume is entrained into the active layer, and the buoyancy decreases through mixing. Hence, even though the problem is now closed in smooth parts of the flow, it is not closed at shocks, where one piece of information is missing. This contrasts with the situation in one layer of homogeneous shallow water, where b is constant, and the energy Equation (3) is used only to diagnose the energy e lost to turbulence at shocks.

For completeness, the simplest form of the equations describing smooth parts of the flow are

$$\begin{aligned} b_t + ub_x &= 0 \\ h_t + (hu)_x &= 0 \\ u_t + uu_x + bh_x + \frac{h}{2}b_x &= 0. \end{aligned} \tag{16}$$

4. Mixing at shocks: an intuitive closure

Waves described by the smooth evolution equations of the preceding sections will typically overturn and break, giving rise to the formation of shocks. Once this happens, the jump conditions (9), (10), and (11) are not sufficient to determine the dynamics. Across shocks, there are three fundamental conservation laws and four unknowns, therefore, a further condition is needed to close the system. This closure will control the amount of entrainment and mixing at shocks. It is natural to propose as a closure that the system will maximize the amount of entrainment and mixing. This not only will provide a potentially useful upper bound on mixing, but also possibly an accurate estimate of the actual dynamics. Much as with the entropy maximization in thermodynamics, this follows from the irreversibility and the rapid timescales of the mixing process.

We will carry out this maximization in three ways: in this and the following sections, through a simple geometrical characterization of maximally mixing shocks; in the next one, through a constrained optimization problem that maximizes the amount of entrainment; and in the one following, through the introduction of an entropy associated to mixing. We will see afterwards that the three closures are in fact equivalent for internal hydraulic jumps.

We propose that maximal mixing occurs when the speed of a shock of the left-most characteristic family (i.e., one with downstream to the right, as in Figure 1), is the downstream characteristic speed

$$c = u_+ - \sqrt{b_+ h_+}. \tag{17}$$

We note that this "jump condition" does not follow from a conservation law and is therefore different from the situation in more conventional systems, where the number of conservation principles and the number of independent variables match. In these, the speed of the shock follows from the integral formulation of those conservation principles. Shocks that mix, however, have the peculiarity that an important physical process—i.e., mixing—is local to them, thus its dynamics is not encapsulated in the equations ruling the flow elsewhere.

To motivate the closure (17), which specifies directly the speed of the shock in terms of the downstream state, consider the inviscid Burgers equation

$$u_t + u u_x = 0. \tag{18}$$

When solutions to (18) become multivalued and shocks form, their position is determined by invoking an integral formulation of (18), physically valid even when the differential equation itself stops making sense. Typically, one would rewrite (18) in the conservation form

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{19}$$

implying that the area $\int u\, dx$ is conserved by the flow. It follows that the shocks move at the speed

$$c = \frac{[u^2/2]}{[u]} = \frac{u_- + u_+}{2}.$$

Figure 2(a) shows the graphical equal area construction for shock placement that implements area conservation [13]. Alternatively, we might impose that the energy $E = \int u^2\, dx$ should be conserved, and rewrite (18) as

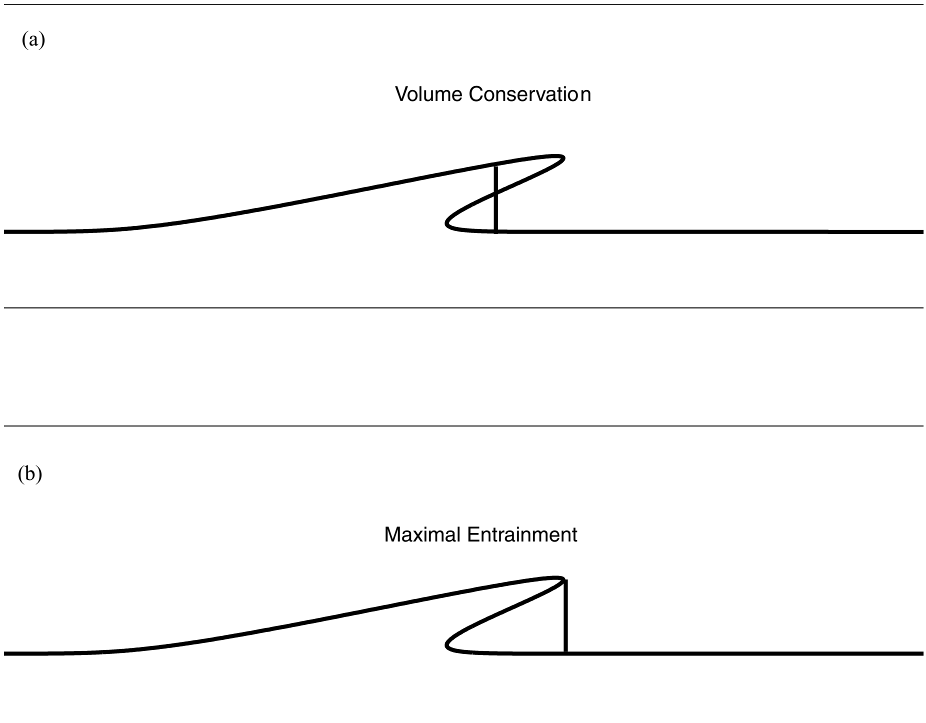

Figure 2. Shock placement for solutions to Burgers. (a) Volume conservation. (b) Maximal entrainment.

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x = 0, \tag{20}$$

with corresponding shock speed

$$c = \frac{[u^3/3]}{\left[\frac{u^2}{2}\right]}.$$

Clearly, there are infinitely many possibilities and only a knowledge of which quantity should be conserved on physical grounds allows us to choose among them. However if our goal is to maximize the entrainment of area, then we should pick the fastest possible shock, with

$$c = u_+, \tag{21}$$

as shown in Figure 2(b). The Lax condition for shocks [13] prevents c from exceeding u_+.

The closure (17) is the generalization of this argument to the internal breaking waves described by (1), (2), and (3). Here the wave characteristic speeds are given by $u \pm \sqrt{bh}$; the choice of the minus sign in (17) applies to shocks with downstream to the right, as in Figure 1.

Interestingly, the condition (21) can be incorporated in the dynamics of (18) by rewriting Burgers in the conservation form

$$\left(\frac{u^n}{n}\right)_t + \left(\frac{u^{n+1}}{n+1}\right)_x = 0, \tag{22}$$

and taking the limit $n \to \infty$.

5. Consequences and range of validity of the closure

We now return to the internal wave breaking problem. Notice that the closure (17) together with the conservations of mass and momentum (Equations (1) and (2)), closes the shock conditions, so that the conservation of energy (3) serves as a diagnostic for the amount of turbulent energy generated at a shock. We will show below that our closure predicts a *positive* generation of turbulent energy for standing shocks, while for moving shocks the condition that turbulence be generated at the shock provides an extra constraint that we argue forms a natural separation between hydraulic jumps and bores.

First, we completely determine the states that may be connected through mixing hydraulic jumps. In particular we show that these states satisfy several intuitively plausible physical properties: that, across jumps, the density decreases, the layer thickness expands, and the velocity and the Froude number decrease.

We introduce the notation

$$B = \frac{b_-}{b_+}, \quad U = \frac{u_-}{u_+}, \quad H = \frac{h_-}{h_+}, \quad F_+ = \frac{u_+}{\sqrt{b_+h_+}}, \quad F_- = \frac{u_-}{\sqrt{b_-h_-}}.$$

Equations (9), (10), and (17) then yield:

$$1 - BHU = \left(1 - \frac{1}{F_+}\right)(1 - BH) \tag{23}$$

$$1 - HU^2 + \frac{1 - BH^2}{2F_+^2} = \left(1 - \frac{1}{F_+}\right)(1 - HU). \tag{24}$$

The product BH, on the other hand, can be written as

$$BH = U^2 \frac{F_+^2}{F_-^2}. \tag{25}$$

This downstream Froude number is directly related to the speed of the shock because

$$c = u_+ - \sqrt{b_+h_+} = \sqrt{b_+h_+}(F_+ - 1). \tag{26}$$

We consider first the case with $F_+ = 1$, corresponding to a standing internal jump ($c = 0$). The solution to the system (Equations (23) and (24)) is

$$U = F_-^{2/3} \geq 1 \tag{27}$$

$$H = \frac{3}{2}\frac{F_-^{2/3}}{F_-^2 + 1/2} \leq 1 \tag{28}$$

$$B = \frac{2}{3}\frac{F_-^2 + 1/2}{F_-^{4/3}} \geq 1. \tag{29}$$

The net rate of production of turbulent energy at the jump is given by

$$[heu] = -\left[\frac{1}{2}hu^3 + bh^2u\right] = \frac{3}{2}b_+h_+^2u_+\left(F_-^{2/3}\frac{F_-^2 + 2}{2F_-^2 + 1} - 1\right) \geq 0. \tag{30}$$

More general values of the downstream Froude number F_+ correspond to shocks moving with respect to the ambient fluid. Using Equation (23) together with the definition of the upstream Froude number F_-, yields the cubic equation for U

$$U^3 - \left(1 - \frac{1}{F_+}\right)U^2 - \frac{F_-^2}{F_+^3} = 0. \tag{31}$$

Again, we think of the downstream information as encapsulated in the Froude number F_+, while the upstream state is completely known, because all three characteristics from upstream reach the shock. Then Equation (31) determines U, and we write all other variables in terms of U and F_+. Equation (31) can be rewritten in the form

$$F_-^2 = F_+^2U^2(1 + F_+(U - 1)), \tag{32}$$

yielding explicit expressions for H and B:

$$H = \frac{(1 + 2F_+)(1 + F_+(U - 1))}{1 + 2UF_+(1 + F_+(U - 1))^2} \tag{33}$$

$$B = \frac{U^2F_+^2}{HF_-^2} = \frac{1 + 2UF_+(1 + F_+(U - 1))^2}{(1 + F_+(U - 1))^2(1 + 2F_+)}. \tag{34}$$

The equations above fully determine all parameters of a shock with given upstream state and downstream Froude number, satisfying the maximal mixing condition (17). We now determine the range of parameters where the shock satisfies the condition that there is a positive conversion of large scale energy into turbulence (i.e., large scale energy dissipation).

First, we compute the turbulent energy generated at a jump:

$$\begin{aligned}
\Delta E &= -(u_+ - \sqrt{b_+ h_+})[eh] + [hue] \\
&= (u_+ - \sqrt{b_+ h_+})\left[\frac{1}{2}hu^2 + \frac{1}{2}bh^2\right] - \left[\frac{1}{2}hu^3 + bh^2 u\right] \\
&= \frac{1}{2}b_+ h_+^2 u_+ \left[\left(F_+ + 1 + \frac{1}{F_+}\right)\right. \\
&\quad \left. - BH^2\left(F_-^2(U-1) + (2U-1) + \frac{1+F_-^2}{F_+}\right)\right].
\end{aligned}$$

Physically, a shock must dissipate large-scale energy and convert it to small scale (turbulent) energy. From the above expression, and using the Equations (32), (33), and (34), this dissipated energy can be expressed in terms of U and F_+ alone. After considerable algebra one can show that positive dissipation occurs when

$$U > \frac{F_+ - 1}{2F_+ + 1} + \frac{\sqrt{\left(F_+^2 - F_+\right)^2 + 3(2F_+ + 1)}}{2F_+^2 + F_+}. \tag{35}$$

Figure 3 displays the admissible domain, which we define as the one where dissipation is positive. We restrict the domain to $U > 1$ (the flow decelerates at the shock), which we show below to hold.

Figure 3 also shows the curve corresponding to the flow upstream being critical, $F_- = 1$. To the right of this curve, and in particular in all the admissible domain, $F_- > 1$. Hence all admissible shocks are supercritical upstream. Standing shocks (with $F_+ = 1$) all lie in the admissible domain. Notice also that in the admissible region, $F_+ > 0$. We will restrict our discussion to the more physically relevant range $0 < F_+ \leq 1$, i.e., flows that are subcritical downstream of the jump.

We now prove a number of physically reasonable consequences of the closure within the admissibility domain: that $U > 1$ (flow deceleration at the shock), $H < 1$ (layer expansion) and $B > 1$ (positive mixing).

The fact that $U > 1$ follows from (31), with $F_- > 1$ and $F_+ \leq 1$, which yield the bounds

$$1 \leq \left(\frac{F_-}{F_+}\right)^{\frac{2}{3}} \leq U \leq \frac{F_-}{F_+}.$$

One shows that $H < 1$ with the following steps: First, showing that, as one changes the parameters F_- and F_+ smoothly within the allowed range, there is only one positive solution to (31). This, together with the fact that the denominator in (33) never vanishes, implies that H, as given by (33), is a continuous function of F_- and F_+. From our special solution for standing

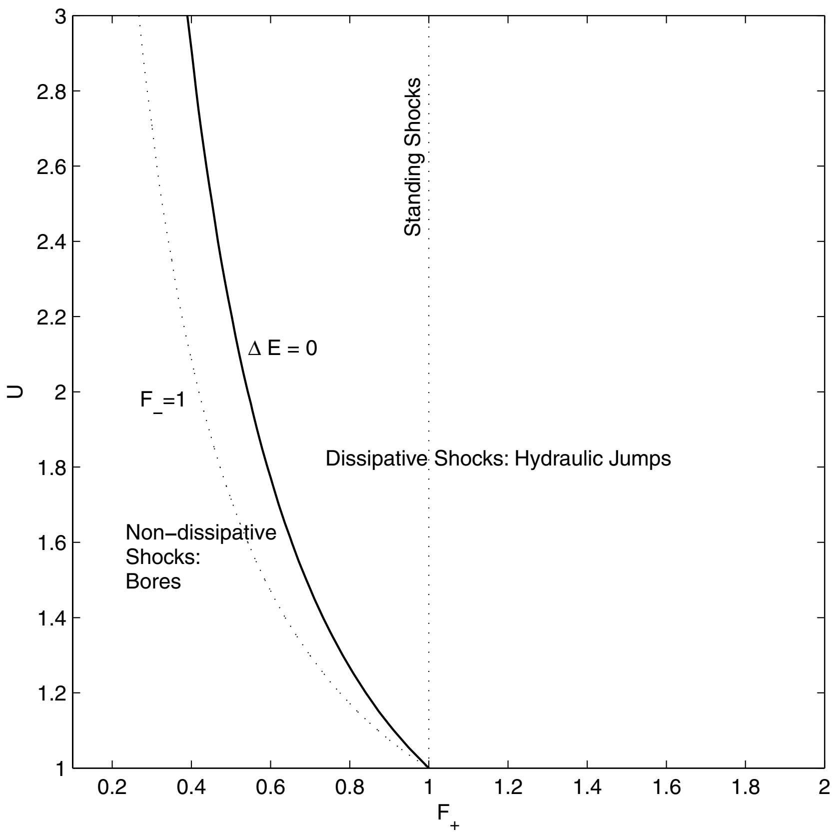

Figure 3. Admissible domain for mixing shocks. Shocks moving at the downtream characteristic speed are only physical if they dissipate energy; the admissible range of parameters for these lies to the right of the curve $\Delta E = 0$. Notice that this admissible range always has a supercritical upstream state. The maximal speed closure is not valid to the left of this curve, where the shocks change their nature from internal hydraulic jumps to internal bores.

shocks, there is at least one shock where $H < 1$. If, changing F_- and F_+ smoothly, one should reach another shock where $H > 1$, then in between there must be one with $H = 1$. However this cannot hold, because $H = 1 \Rightarrow U = 1$ from (33), while we have just proved that U must be bigger than one.

To show that $B > 1$, we proceed along similar lines: From (34), we see that B is a continuous function of the parameters. Setting $B = 1$ in (34) yields $U = 1$ as the only permissible solution. Therefore, by continuation from the case of a standing shock, we conclude that $B > 1$.

To the left of the boundary between dissipative and nondissipative shocks in Figure 3 our closure becomes unphysical, with spontaneous energy generation at the shocks. One solution in this regime, is to replace the closure proposed here by a zero energy dissipation closure [11]. The two closures connect smoothly across the zero-dissipation boundary.

Hence we have distinguished two kinds of mixing internal shocks: those that dissipate energy, and move at their maximal allowable shock speed, as given by the Lax condition, and those that cannot reach this maximal speed, and instead are characterized by being nondissipative. This provides in fact a precise, natural distinction between internal *hydraulic jumps* and *bores* [4].

In river flow terminology, hydraulic jumps are standing shocks, typically occurring immediately downstream of spillways, waterfalls or submerged obstacles, while bores are rapidly moving flood waves, typically arising from high tides downstream or high rains or intensive ice melting upstream. From the mathematical viewpoint, at least in the simplest, frictionless models, the two kinds of discontinuities are completely equivalent, because a Galilean transformation maps one into the other.

For internal breaking waves in a one-and-a-half layer model, however, there is a distinguished frame of reference: the one for which the ambient fluid is at rest. As a consequence, internal hydraulic jumps and bores do exhibit fundamentally different behavior. It was first pointed out in [4] that gravity currents are limiting cases of internal bores. Their characteristic pronounced heads, with entrainment though their backs, are strikingly different from the profiles of internal hydraulic jumps. The latter, reminiscent of the external hydraulic jumps at spillways, entrain ambient fluid through the wave's front. Figure 4 displays the two typical profiles, together with the flow surrounding them. It was also pointed out in [4] that

- the vorticity associated with the shear has opposite signs for hydraulic jumps and bores, resulting in very different entrainment properties; and that
- the active layers of internal hydraulic jumps dissipate energy, while those of internal bores are energy conserving.

This latter dissipative contrast, which was exploited in [4] to model jumps and bores differently, is exactly the distinction that our model makes when the active constraint switches from the Lax entropy condition to the nonnegativity of the energy dissipation. Hence, in our model, the characterization of a discontinuity as a hydraulic jump or a bore is sharp, and internal bores have maximal mixing efficiency, because all available energy is used for mixing. Hydraulic jumps are necessarily supercritical upstream, while we see in Section 9 that internal bores constitute the natural occurrence in subcritical flows.

6. Maximal entrainment

The motivation for the closure above, that shocks move at their maximal possible speed, is the presumption that the mixing rate increases with the speed of the shock. This is intuitively plausible, yet needs to be shown to

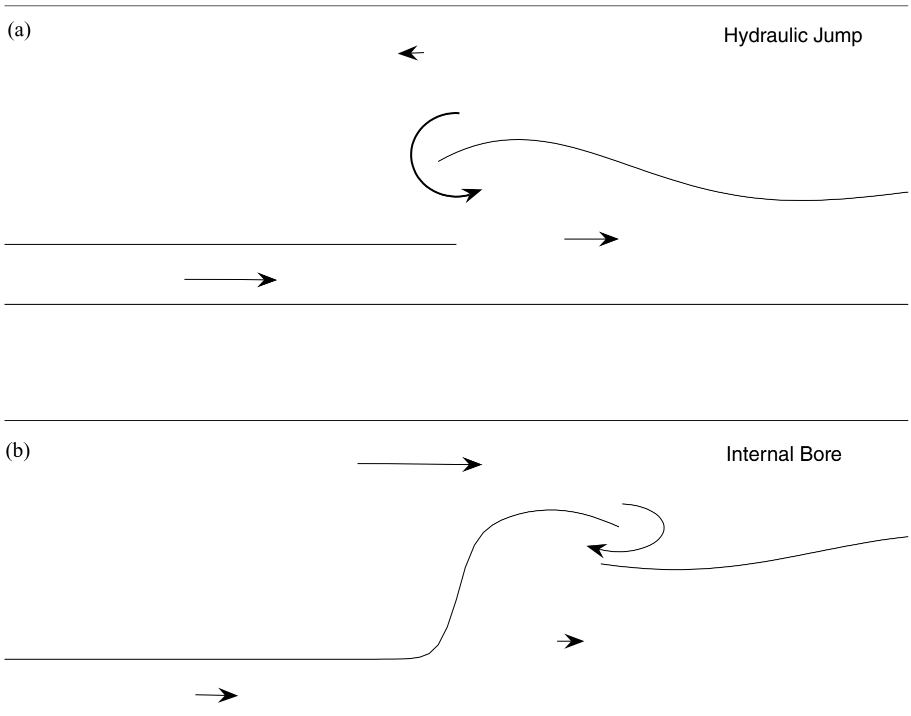

Figure 4. Flow through internal hydraulic jumps and bores, in frames of reference attached to the shock waves. The nature of entrainment varies widely from one to the other, giving rise to very different characteristic profiles. In a model that maximizes entrainment, the two can be distinguished based on whether the active constraint is energy conservation (for bores) or the Lax entropy condition (for hydraulic jumps).

hold. In this section, we see that the rate of entrainment of ambient fluid into the active layer is indeed maximized by shocks moving at the downstream characteristic speed. The following section will place this result in a more universal framework, where the rate to maximize is not that of entrainment, but of production of a suitable entropy associated to mixing.

The rate ϵ of entrainment at a shock moving at speed c equals the increase of volume flow across the shock:

$$\epsilon = [hu] - c[h].$$

Our problem consists of choosing the speed c so as to maximize this mixing rate, subject to the constraints of mass and momentum conservation (Equations (9) and (10)) and the Lax entropy conditions

$$u_+ - \sqrt{b_+ h_+} \le c \le u_- - \sqrt{b_- h_-}.$$

The values of all variables upstream are given, because they arrive to the jump along characteristics. Only the leftmost-going characteristic brings information from downstream, in the form of a differential constraint:

$$\frac{1}{2}\frac{dB}{B} + \frac{dH}{H} - F_+\frac{dU}{U} = 0, \tag{36}$$

as follows from (16). It will turn out, however, that the form of this constraint is immaterial for proving the local optimality of our closure (not so for a global result, that we postpone to the following section.)

For convenience, we nondimensionalize the entrainment rate ϵ and the shock speed c using the upstream volume flux and fluid speed respectively; i.e., we define $E = \frac{\epsilon}{u_- h_-}$ and $C = \frac{c}{u_-}$. In terms of these, the problem becomes

$$\text{maximize } E = \frac{1}{HU}[(1 - HU) + CU(H - 1)],$$

subject to

$$1 - BHU = CU(1 - BH), \tag{37}$$

$$1 - HU^2 + \frac{1 - BH^2}{2F_+^2} = CU(1 - HU), \tag{38}$$

$$BH = U^2\frac{F_+^2}{F_-^2}, \tag{39}$$

$$\frac{1}{U}\left(1 - \frac{1}{F_+}\right) \leq C \leq 1 - \frac{1}{F_-}, \tag{40}$$

and (36), with the Froude numbers satisfying $0 \leq F_+ \leq F_-$ and $F_- \geq 1$. (Equations (37) and (38) represent mass and momentum conservation, (39) follows from the definition of the Froude numbers, and (40) are the Lax entropy conditions.) Replacing BH from (39) into (37) yields the following cubic equation for U in terms of C:

$$(1 - C)U^3 + \left(\frac{F_-}{F_+}\right)^2 (CU - 1) = 0, \tag{41}$$

while the same replacement in (38) yields the expression

$$HU^2 = \frac{1 - CU + \dfrac{1}{2F_+^2}}{1 - C + \dfrac{1}{2F_-^2}}.$$

With this, our problem reduces to

$$\text{maximize } E = (1-C)\left[-1 + \frac{1-CU}{1-C}\frac{1-C+\dfrac{1}{2F_-^2}}{1-CU+\dfrac{1}{2F_+^2}}U\right], \tag{42}$$

subject to (41), (40), and (36).

We prove now that a local maximum of the normalized entrainment E is achieved when C is minimal; i.e., it is given by the downstream characteristic speed

$$C = \frac{1}{U}\left(1 - \frac{1}{F_+}\right). \tag{43}$$

To see this, notice that (41) defines U as a function of C and F_+. Then the entrainment E in (42) is also a function $E(F_+, C)$, shown in Figure 5. Straightforward differentiation shows that, as can be seen in the figure, at the closure (43),

$$\frac{\partial E}{\partial F_+} = 0.$$

Next we prove that, whenever E is positive or zero, $\frac{\partial E}{\partial C} \leq 0$. Because $C \leq 1$, as follows from the Lax condition (40), and $E \geq 0$ by hypothesis, we have that

$$\frac{\partial E}{\partial C} \leq \frac{\partial}{\partial C}\left[\frac{1-CU}{1-C}\frac{1-C+\dfrac{1}{2F_-^2}}{1-CU+\dfrac{1}{2F_+^2}}U\right].$$

Expanding this derivative, using $\frac{\partial U}{\partial C} = \frac{U(1-U)}{(1-C)(3-2CU)}$ yields

$$\frac{\partial E}{\partial C} \leq \frac{3U(1-CU)(1-U)}{4F_-^2F_+^2(1-C)^2(3-2CU)\left(1-CU+\dfrac{1}{2F_+^2}\right)^2} \leq 0,$$

where we have used the facts that $U \geq 1$ (the flow slows down across the shock) and $CU \leq 1$ (as follows from (41)).

This completes the proof of local optimality of the closure (43), because $\frac{\partial E}{\partial F_+} = 0$, $\frac{\partial E}{\partial C} \leq 0$, and $dC \geq 0$, and hence

$$dE = \frac{\partial E}{\partial C}dC \leq 0.$$

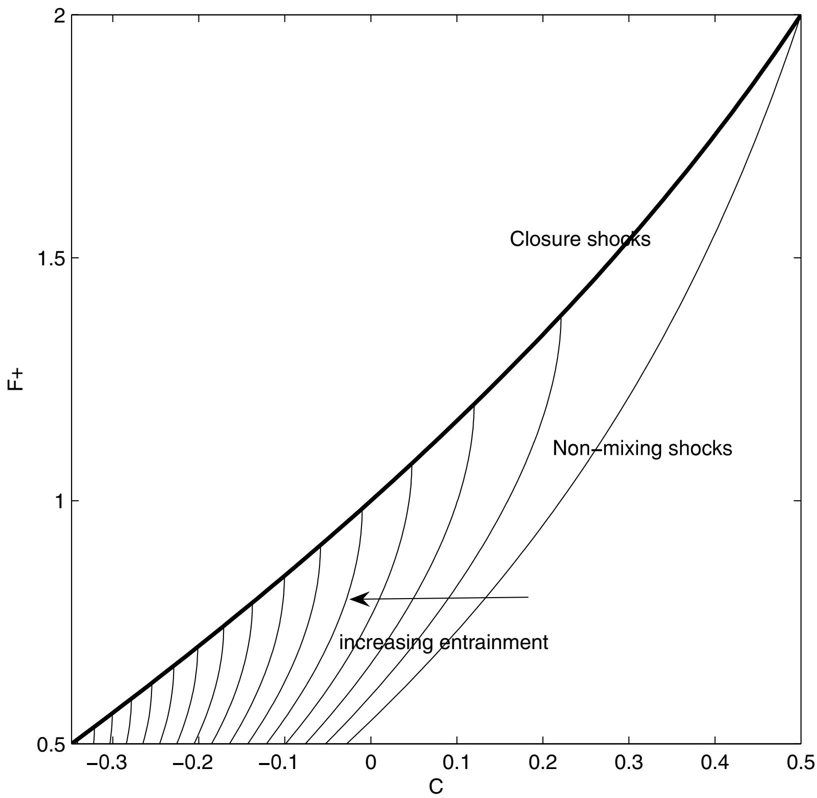

Figure 5. Contours of constant entrainment E as functions of the downstream state, characterized by its Froude number, and the speed C of the shock. The maximal shock speed allowed by the Lax condition is the upper curve, while the lower curve corresponds to nonmixing hydraulic jumps, identical to the ones in one layer shallow water. Notice that the contours become vertical as they reach the upper curve, indicating that, locally, the entrainment does not depend on F_+. We prove below that the upper curve gives a global maximum of entrainment when C is varied, consistently with the constraint given by the downstream characteristic entering the shock.

7. A general result for incomplete systems of conservation laws

The proof in the prior section can be made far more general, and applied to different physical settings. Starting with a system of n conservation laws

$$u_t + F(u)_x = 0, \quad u = \begin{pmatrix} u_1 \\ \cdots \\ u_{n-1} \\ u_n \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ \cdots \\ F_{n-1} \\ F_n \end{pmatrix}, \tag{44}$$

where only the first $n-1$ laws apply at shocks, we consider a shock wave moving into a known upstream state u_-. In regular systems, this shock will be completely specified by a single piece of information coming from downstream. In our case, however, because one jump condition is missing, the shock speed c is another free parameter.

There is no production of the $n-1$ quantities conserved at the shock, so we have

$$0 = -[u_j]\,dc + e_j^t(A - cI)\,du_+, \quad j \in (1, \ldots, n-1), \tag{45}$$

where e_j is the canonical unit vector with a one in position j and zeros elsewhere, and A is the matrix with entries

$$A_{ij} = \frac{\partial F_i}{\partial u_j}(u_+).$$

The production p of the quantity u_n at the shock is given by

$$p = -c[u_n] + [F_n], \tag{46}$$

with variation

$$dp = -[u_n]\,dc + \sum_{j=1}^{n} \frac{\partial F_n}{\partial u_j}(u_+)\,du_j^+ - c\,du_n^+ = -[u_n]\,dc + e_n^t(A - cI)\,du_+. \tag{47}$$

Consider now an eigenvalue μ of A (a characteristic speed of the system), with corresponding left eigenvector

$$l = (l_1, \ldots, l_{n-1}, 1). \tag{48}$$

Multiplying the system (45) and (47) by l on the left, we obtain that

$$dp = -\langle l, [u]\rangle\,dc + (\mu - c)\langle l, du_+\rangle. \tag{49}$$

In particular, when the shock speed c agrees with the characteristic speed μ,

$$dp = -\langle l, [u]\rangle\,dc, \tag{50}$$

showing that the rate p of production of u_n at the shock depends locally only on the speed c. When μ is the minimal (left-most) characteristic speed, and $\langle l, [u]\rangle > 0$, the corresponding shock speed $c = \mu$ yields a local maximum for the production rate p.

In fact, more is true: the information that arrives at the shock from downstream along the μ-characteristic is precisely of the form

$$\langle l, du_+\rangle = 0. \tag{51}$$

So even away from $c = \mu$, the condition

$$dp = -\langle l, [u]\rangle\,dc \tag{52}$$

holds when the downstream state is constrained by the information carried by the left-most moving characteristic. Hence, if we can show that $\langle l, [u]\rangle > 0$, then the *global* maximum of p is achieved at the minimal c, which is either the one given by the Lax condition, or by another constraint, such as the positive dissipation of energy (35), if it turns active first.

This condition is indeed satisfied in mixing hydraulics, as the following argument shows. The conserved variables are $(q = bh, m = hu, h)$, satisfying the equations

$$\begin{aligned}(q)_t + \left(\frac{qm}{h}\right)_x &= 0\\ (m)_t + \left(\frac{m^2}{h} + \frac{qh}{2}\right)_x &= 0\\ h_t + m_x &= 0,\end{aligned}$$

which are (Equations (1), (2), (13)) rewritten in terms of the conserved variables. Only the first two conservation equations hold at shocks. The left eigenvector corresponding to the left-most moving characteristic,

$$\frac{dx}{dt} = \frac{m}{h} - \sqrt{q},$$

is given by

$$l = \left(\frac{h}{(1+2F)q}, \frac{2Fh}{(1+2F)m}, 1\right).$$

Hence we need to show that

$$-\frac{dp}{dc} = \langle l, [u]\rangle = [h] + \frac{h_+}{(1+2F_+)q_+}[q] - \frac{2F_+h_+}{(1+2F_+)m_+}[m] \geq 0.$$

Multiplying by $\frac{1+2F_+}{h_+}$, a positive number, we obtain

$$\frac{1+2F_+}{h_+}\langle l, [u]\rangle = 2 - H(1+B) + 2F_+H(U-1).$$

Because $U > 1$, it is enough to show that

$$H(1+B) \leq 2.$$

If $HB < 1$, the statement holds, because $H < 1$. If not, Equation (37) implies that $C \geq 0$. From the positivity of entrainment at the jump,

$$1 - HU \geq CU(1-H),$$

we obtain that

$$H(1+B) \leq \frac{2(1-CU)}{U(1-C)} \leq \frac{2}{U} \leq 2.$$

Hence the rate of entrainment decreases monotonically with the speed c of the shock, and the global maximizer of entrainment has, among all allowed speeds, the one moving fastest to the left. This maximal speed is given either by the closure (17), or by the constraint of positive energy dissipation, if it acts first.

8. Maximizing a mixing entropy

Entrainment is a meaningful way of quantifying mixing in two layer flows. In more general settings, however, we need a way of measuring mixing that does not depend on a particular scenario. Breaking internal waves in a continuously stratified flow, for instance, would lead to mixing events that cannot be easily described in terms of entrainment. More generally, small-scale processes not captured in detail by a particular model of fluid motion, typically lead to mixing, through a variety of processes commonly placed under the umbrella of turbulent diffusion.

To apply the ideas of this paper to such general scenarios, we need to quantify the degree of mixing of a flow in a more universal way. Then the [incomplete] set of equations modeling the dynamics can be treated as constraints in an optimization procedure that seeks to maximize this mixing measure. An entropy associated to the mixing of two fluids has been proposed in [14]. Based on standard arguments in statistical physics, it adopts the form

$$S = -\int (a\log(a) + (1-a)\log(1-a))\, dV, \tag{53}$$

where $a(x, t)$ is the volume fraction of one of the two fluids in position x at time t.

In our entraining hydraulic jump, the two fluids mixing are the ambient fluid and the fluid in the active layer arriving at the shock from upstream. The corresponding vertically integrated entropy vanishes upstream of the jump, and downstream it equals

$$h_+ s(1/B), \quad \text{with} \quad s(a) = -[a\log(a) + (1-a)\log(1-a)]. \tag{54}$$

This definition would suffice to model a single shock. Yet, if one seeks to model the whole flow through a constrained optimization approach, one may think of the active layer as having originated in its entirety from the mixture of two fluids: an ambient fluid of density ρ_a and some denser fluid, of reference density ρ_0, satisfying $\rho_0 \geq \rho(x, t)$. Then vertically integrating the entropy in (53) yields

$$S = hs(b_*), \tag{55}$$

and

$$b_* = \frac{\rho - \rho_a}{\rho_0 - \rho_a} = \frac{\rho_a}{g(\rho_0 - \rho_a)} b. \tag{56}$$

It was shown in [11] that all possible entropies for the system (16), in the sense given to the term in systems of conservation laws [15], are of the form

$$S = hf(b, e). \tag{57}$$

These are conserved locally in smooth parts of the flow:

$$S_t + (uS)_x = 0. \tag{58}$$

In addition, convex functions of b alone, $f(b, e) = g(b)$, including the physically motivated choice $g(b) = s(b_*)$, increase at shocks, with corresponding entropy production given by

$$p = (u_+ - c)h_+g_+ - (u_- - c)h_-g_- = (u_- - c)h_-b_- \left(\frac{g_+}{b_+} - \frac{g_-}{b_-} \right) \geq 0, \tag{59}$$

where we have used the buoyancy conservation Equation (9). (The positivity of p follows from the convexity condition which implies that $\frac{g(b)}{b}$ is a decreasing function, together with the positive entrainment condition $b_+ \leq b_-$.)

Moreover, in view of the result of the prior section, that entrainment is maximized for shocks that move at the minimal speed $c = u_- - \sqrt{b_-h_-}$, we now show that the entropy production p in (59) for any convex specific entropy $g(b)$ is also maximized by these shocks. To see this, it is convenient to rewrite (59) in the following way:

$$p(q, \epsilon) = q(g(b_+) - g(b_-)) + \epsilon(g(b_+) - g(b_a)). \tag{60}$$

Here $q = (u_- - c)h_-$ is the volume flow from upstream into the jump and ϵ the entrainment of ambient fluid, both decreasing functions of c. We have included the ambient entropy $g(b_a)$ for generality, even though it is zero for our specific application. We also have that

$$b_+(q, \epsilon) = \frac{qb_- + \epsilon b_a}{q + \epsilon}. \tag{61}$$

The production p in (60) is a growing function of both q and e (that is, the more mixing, whether by entrainment or by flux into the shock, the more entropy), and consequently a decreasing function of c. To show this, from the symmetry of (60), it is enough to show that p is a growing function of q. This follows simply from convexity: adding a positive Δq to q yields

$$\begin{aligned} p(q + \Delta q, \epsilon) - p(q, \epsilon) &= (q + \epsilon + \Delta q)g(b_+(q + \Delta q, \epsilon)) \\ &\quad - (q + \epsilon)g(b_+(q, \epsilon)) - \Delta q g(b_-) \\ &\geq 0, \end{aligned}$$

where the inequality follows from

$$
\begin{aligned}
(q+\epsilon+\Delta q)g(b_+(q+\Delta q,\epsilon)) &= (q+\epsilon+\Delta q)g\left(\frac{(q+\Delta q)b_- + \epsilon b_a}{q+\epsilon+\Delta q}\right)\\
&= (q+\epsilon+\Delta q)g\left(\frac{(qb_- + \epsilon b_a) + \Delta q b_-}{q+\epsilon+\Delta q}\right)\\
&= (q+\epsilon+\Delta q)g\left(\frac{(q+\epsilon)b_+(q,\epsilon) + \Delta q b_-}{(q+\epsilon)+\Delta q}\right)\\
&\geq (q+\epsilon)g(b_+(q,\epsilon)) + \Delta q g(b_-).
\end{aligned}
$$

9. A global optimization approach to the dynamics of mixing

We have so far studied the properties of the shock wave that maximizes entrainment, or entropy production, subject to the constraints brought along characteristics from upstream and downstream. We think of the jump conditions associated to conservation of volume, mass and momentum, as well as the Lax entropy conditions and the requirement of positive energy dissipation, as extra constraints imposed upon this optimization problem.

In this section, we investigate whether this approach can be extended to the description of the whole flow: instead of applying an extra closure (nonmixing) wherever the solution is smooth, and modeling the shocks through the procedure above, to think of the [incomplete] dynamical equations as a set of constrains, and maximize mixing globally.

It is convenient to describe the dynamics in terms of the conserved quantities $q = bh$, and $m = hu$, in addition to the height h itself. The equations for conservation of buoyancy and momentum, positive entrainment and positive energy dissipation read

$$q_t + \left(\frac{qm}{h}\right)_x = 0 \tag{62}$$

$$m_t + \left(\frac{m^2}{h} + \frac{1}{2}qh\right)_x = 0 \tag{63}$$

$$h_t + m_x \geq 0 \tag{64}$$

$$\left(\frac{m^2}{h} + qh\right)_t + \left(\frac{m^3}{h^2} + 2qm\right)_x \leq 0. \tag{65}$$

The quantity we would like to maximize is the global entropy production,

$$\text{maximize}\frac{d}{dt}\int hs(b)\,dx. \tag{66}$$

In smooth parts of the flow, Equations (62), (63), and (65) can be combined to yield

$$(1 - F^2)(h_t + m_x) \leq 0, \tag{67}$$

where F is the Froude number, $F = \frac{m}{h\sqrt{q}}$. Two different situations arise depending on wheather the flow is subcritical ($|F| < 1$) or supercritical. If the flow is subcritical, (67) and (64) yield

$$h_t + m_x = 0, \tag{68}$$

that is, no entrainment takes place in smooth parts of the flow (see Section 3), in agreement with our closure assumption. As the waves of the system nonlinearly deform and break, they will give rise to entraining shocks, that will maximize entropy production by moving at their maximal allowed speed, as we have shown above. Yet these are shock waves that are subcritical on both sides, so the active constraint will be that of energy conservation. Therefore,

Globally maximally mixing subcritical flows are described precisely by our closure above: no mixing takes place in smooth parts of the flow, and when waves break, they form energy preserving bores.

These flows can be computed in two alternative ways: through the global optimization procedure given by (62)–(66), or, more simply, through the solution of a complete set of conservation laws, given by (62) and (63) and the energy dissipation Equation (65) turned into an equality. This model has various potentially significant applications, including the ocean's upper mixed layer, as it grows by entraining water from below through breaking waves at the interface. A sample computation modeling a gravity current is shown in Figure 6.

The supercritical case is more subtle: Equations (67) and (64) are now the same, and they do not impose any constraint on the maximization procedure, that will gain from any increase in $h_t + m_x$. This is a manifestation of the fact that *supercritical flows are unstable to mixing*: for given values of q, m and $E = \frac{m^2}{h} + qh$, there are two values of h:

$$h = \frac{E \pm \sqrt{E^2 - 4qm^2}}{2q}, \tag{69}$$

one super and one subcritical. The latter one, with the larger h and smaller $b = q/h$, yields a more mixed state. More precisely, any height h in between the two solutions (69), together with the definitions $b = q/h$ and $u = m/h$, yields a state with total energy smaller than E. Hence it is possible for a supercritical state to liberate energy while mixing.

Despite their natural instability, supercritical flows, and the subsequent mixing events that turn them subcritical, do occur in nature [16]. They originate

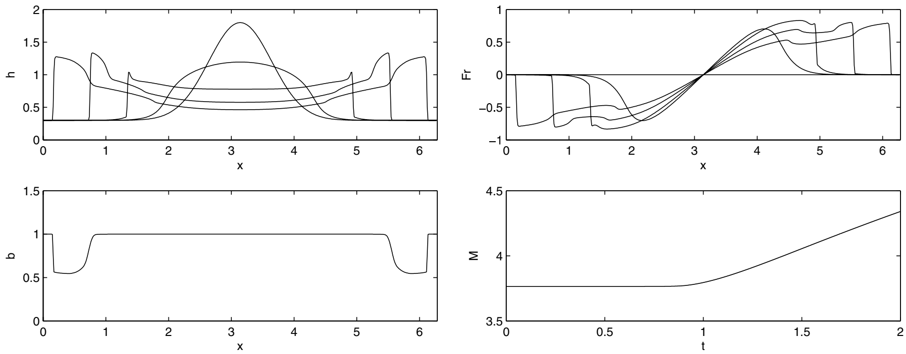

Figure 6. Numerical solution of a subcritical entraining flow: release of a bulge of heavy fluid in a lighter ambient. The plot on the top-left shows the evolution of the interface, with a bore forming. The bottom-left plot shows the density profile at the last computed time: once a bore forms, entrainment of ambient fluid occurs, lowering the density. The top-right plot displays the evolution of the [signed] Froude number, which remains subcritical throughout the run. The bottom-right plot shows the total volume in the active layer as a function of time; it starts increasing once the bore forms and entrainment occurs.

when dense currents flow over sills, where they switch from subcritical to supercritical, in what oceanographers denote *hydraulic control* [2]. Examples include the Mediterranean outflow over Gibraltar, the dense overflow of Arctic water over the Denmark strait, that of Antarctic water through the Filchner depression, and the Antarctic bottom water flowing into the Brazilian basin through the Vema channel. Thus the properties of much of the abyssal water in the world's oceans is determined by the rate of entrainment in supercritical flows.

How do we conciliate the existence of supercritical flows in nature, with their instability to mixing, and with our modeling proposal of maximizing a mixing entropy? It is all a matter of time scales. The intuition behind the modeling hypothesis of maximizing the degree of mixing of a flow is that, given enough time, a fluid will mix as much as global constraints, such as conservation of energy and momentum, permit. Yet mixing of a stratified fluid is not an instantaneous phenomenon, and the supercritical sections of geophysical flows, typically between a sill and a hydraulic jump, are short, and travelled rapidly by the fluid. Hence there is only time for partial mixing prior to the jump; the role of the latter is in fact to complete the mixing process. Observations of the Mediterranean outflow [17] show strongly localized entrainment and mixing at the Portimão canyon, where the flow appears to transition from super to subcritical. Partial mixing is observed throughout the supercritical stretch of the flow. Experiments on internal hydraulic jumps [18] also show that substantial interfacial entrainment of ambient fluid into the supercritical plume occurs before the jump, consistently with the flow's instability to mixing.

10. Conclusions

Internal breaking waves at the interface between two miscible fluids lead to the entrainment and subsequent mixing of an a priori unknown amount of one fluid mass into the other. In this work, we have developed a closure for these waves based on the hypothesis that they will maximize the rate of mixing, subject to the dynamical constraints provided by conservation laws (of mass, momentum and energy) and causality (the Lax entropy conditions, nonnegative turbulence production). We have found that entrainment is maximized when the speed of the shock against the velocity upstream is fastest. Depending on whether the active constraint limiting this speed is given by the Lax entropy conditions, or by the requirement that the energy in the mean flow converted into turbulence be nonnegative, two types of shocks arise, corresponding to internal hydraulic jumps and bores respectively.

We have shown the equivalence of two natural ways of measuring mixing: by the entrainment rate, and by the rate of a suitably defined entropy production. The latter has the advantage of being easily generalizable beyond the one-and-a-half layer scenario of this work, to arbitrary flows. Moreover, it permits replacing the local maximization at shocks by a global optimization procedure, whereby the production rate of the total entropy content of the flow is maximized, subject to the coarse-grained dynamics provided by a large-scale model.

In the context of subcritical flows in the one-and-a-half layer model, such global procedure leads to a dynamical system that conserves not only mass and momentum, but also the energy content of the mean flow (i.e., it does not produce any turbulence). Under these constraints, no entrainment takes place away from the shocks. These shocks are faster than their single layer counterpart, because they do not preserve the mass content of the active layer (which grows due to the entrainment process), but its energy (which is partially dissipated in single layer jumps). Supercritical flows, by contrast, are unstable to mixing. The results of this paper are still applicable at the hydraulic jump itself, but the detailed dynamics of the flow nearby, including interfacial mixing driven by shear instability, cannot be resolved by our model.

Some of the results of this work could be applied to general scenarios beyond stratified flows, whenever a system of conservation laws is "incomplete", leading to a closure problem at shocks. This is the case, for instance, of detonation waves in combustion theory, which again move at the maximal speed consistent with the Lax entropy conditions, and maximize entropy production (the conventional entropy of gas dynamics this time, not the one associated to fluid mixing). This analogy helps develop a physical intuition on these flows (the instability of supercritical flows to mixing, for instance, is the natural analogue of the instability to combustion of the gas mixture prior to the detonation), and provides a general mathematical framework where questions may be formulated more easily, and find a broader set of applicability.

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