RADIAL STIFFNESS

OF A

BICYCLE WHEEL

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AN ANALYTICAL STUDY

José María Mínguez (corresponding author)
Dpto. de Física Aplicada II
Facultad de Ciencia y Tecnología
Universidad del País Vasco
Aptdo. 644
48080 Bilbao
SPAIN
Email: josemaria.minguez@ehu.es

Jeffrey Vogwell
Department of Mechanical Engineering
Faculty of Engineering and Design
University of Bath
U. K.

Abstract

A theoretical analysis is carried out to study the behaviour of the structural components of a spoked bicycle wheel when radially loaded. This has been done so as to establish convenient mathematical relationships for quantifying the individual contributions which the spokes and the rim make to the radial stiffness of the wheel. The effect of spoke pretension is considered and also the influence that this has on the efficient distribution of load and upon the strength of the wheel components.

Keywords: radial stiffness, bicycle wheel, spokes, rim, pretension

Introduction

Although the wheel has been around for millennia and the bicycle wheel for over a century, there are still radical changes being made to the design of the modern spoked bicycle wheel (Chandler, 2002). Racing cycle wheels, as used in the Tour de France for example, have changed substantially from the multi-spoke wheels with open cross section wheel rims as used on the traditional racing bike (Schraner, 1999; Okajima et al. 2000; Muraoka et al. 2001). Leading bicycle wheel manufacturers, such as Mavic and Shimano, have developed wheels which efficiently use few metal spokes that attach to wheel rims which comprise part closed and open cross sections. Modern materials, such as carbon fibre, and developments in forming techniques have resulted in wheels manufactured as single items, effectively having even fewer spokes, and such wheels are widely used in speed racing on velodrome circuits

The requirements of a modern bicycle wheel are many. The wheel must transfer the weight of the cyclist (via the frame, forks and axle) radially from the wheel hub to the rim and thus to the ground (via the tyre). In order to achieve traction, and thus wheel rotation, torsion must be transmitted from the chain sprocket (attached to the hub) to overcome rolling resistance at the tyre (which fits within the rim). In addition, a wheel must be sufficient strong to withstand shock loads when riding over bumpy terrain and withstand cornering loads (Gordon, 2004). Also the rim must run true and the spokes play a major role in achieving this together with the selection of appropriate materials (McMahon and Graham, 1992).

Understanding how a spoked wheel works is clearly essential but perhaps not immediately obvious as pre-tensioning of the spokes is fundamental to a wheel functioning efficiently, as shown by Dietrich (1993, 1999). Without pre-tightening of the spokes, for example, the weight would only be supported by those spokes in tension as the spokes in the lower half of the wheel would be in compression and, because of the slender shape of a spoke, it means that they will buckle rather than support any appreciable compressive load.

Traditionally wheels have used many spokes and this is because the open section rim lacked rigidity and would distort too much if the spacing between successive spokes were large (Brandt, 1993). However, modern rims are much more rigid in bending and torsion because of the part closed cross-section and so fewer spokes can be used, according to Hed and Haug (1989) and Muraoka et al. (2001). An optimum design, therefore, seeks a compromise between acceptable rim and spoke strength and stiffness (Gordon, 2004).

Although finite element analysis can be used to good effect to model a wheel assembly and also mechanical tests exist confirming performance (Rinard, 2002), there are advantages also in taking an analytical approach (Hull et al., 2002) as the significance of the many variables can more readily be seen. So the primary objective of this paper is to establish an analytical expression for determining the radial stiffness of an ideal spoked wheel in terms of the major defining parameters.

Idealised wheel geometry

The following analysis is based on an ideal spoked wheel as shown in figure 1. The wheel consists of the circumferential rim, the hub and a number of spokes, N which connect as pin joints between the rim and the hub. The rim has a radius, R and the width of the hub is $2d$ and so the angle, γ between the centre axis is given by :

$$\tan\gamma = \frac{d}{R} \qquad (1)$$

To determine the wheel radial stiffness a downward load P is applied at the hub and this is reacted by an equal force, P at the rim where the rim contacts with the ground. The deformation of the wheel is the shortening of the distance between the hub and the contact point of the rim with the floor.

The original length of the spokes when not under load is L and the length when secured at the rim and hub and tightened and thus subjected to pre-tension is $L_o = \dfrac{R}{\cos\gamma}$.

Force equilibrium

Now consider radial force equilibrium for half a wheel whereby all spokes are pre-tensioned with tensile load, T_o and C is the circumferential rim force (see figure 2)..

For N number of spokes (and with the top spoke aligned with vertical datum) the equilibrium condition for the half wheel is

$$2C = T_0 \cos\gamma \sum_{\theta=-\pi/2}^{\pi/2} \cos\theta_i \qquad (2)$$

in which the sum is extended over all the spokes anchored to the half rim. So, if the total number of spokes of the wheel is N, the number of spokes per angle unit is $N/2\pi$ and the sum may be evaluated as follows

$$\sum_{\theta=-\pi/2}^{\pi/2} \cos\theta_i = \int_{-\pi/2}^{\pi/2} \frac{n}{2\pi} \cos\theta d\theta = \frac{N}{\pi} \qquad (3)$$

which is acturate for most practical cases. For example, if there are 16 spokes, the sum has a value of 5.027 whereas the N/π approximation gives 5.093 indicating an error of just 1.3 %.

Consequently, the hoop compression of the rim caused by the pretension of the spokes is

$$C = \frac{NT_0 \cos\gamma}{2\pi} \tag{4}$$

from which the relationship between the extension of the spokes ΔL due to their pretension T_0 and the contraction of the radius ∇R of the wheel rim due the hoop compression can be easily found.

Effectively, L being the original length of a spoke, E_S the modulus of elasticity and A_S its cross section,

$$\Delta L = L_0 - L = \frac{R}{\cos\gamma} - L = \frac{T_0 L}{A_S E_S} \tag{5}$$

and the radius of the wheel rim, with a cross section A_R and a modulus of elasticity E_R, will be reduced by an amount

$$\nabla R = \frac{CR}{A_R E_R} = \frac{NT_0 L \cos^2\gamma}{2\pi A_R E_R} = \frac{N\cos^2\gamma}{2\pi \frac{A_R}{A_S}\frac{E_R}{E_S}}\Delta L \tag{6}$$

As the rim cross sectional area is typically about 30 times greater than the spoke cross sectional area, because the rim is usually made from an aluminium alloy whereas the spokes are from steel, in the case of a wheel having 16 spokes and taking $\cos\gamma \approx 0.97$ means that :

$$\nabla R \approx 0.25\Delta L \tag{7}$$

Equation (7) describes the stiffness of the wheel rim with respect to the spoke tension and indicates that the variation of the radius of the rim and the extension of the spokes are of the same order of magnitude.

However, this reducing of the radius of the rim must not be accounted for when analysing the stiffness of the wheel, for it takes place when the wheel is being built, previously to its use and loading. Nevertheless it was studied to illustrate how the spokes and the rim become a joint structure.

Spoke contribution

To study the radial stiffness of the wheel it will be assumed that the hub supports an in-plane load *P*, which causes a radial displacement *a* of the hub. The aim of this analysis is to establish a relationship between *a* and *P* in order to deduce the stiffness.

In this section only the displacement a_S of the hub resulting from the strain and the deflection of the spokes is going to be calculated. The displacement a_R of the hub due to the distortion of the rim is calculated in the following section and then added to a_S.

Figure 3 shows the hub loaded and displaced a_S at the centre of the rim, with only one spoke shown for clarity but the calculations account for all spokes.

When the load *P* is applied on the hub it is supported by all the spokes, each of them being subjected to a tension T_i and stretched to a length

$$L_i = \frac{R + a_S \cos\theta_i}{\cos\gamma} \tag{8}$$

according to figure 3.

Then the fraction P_i of the total load *P* assumed by each spoke is

$$P_i = T_i \cos\gamma \cos\theta_i \tag{9}$$

the tension of the spoke being worth

$$T_i = \frac{A_S E_S}{L}\left(\frac{R + a_S \cos\theta_i}{\cos\gamma} - L\right) = T_0 + \frac{A_S E_S}{L}\frac{a_S \cos\theta_i}{\cos\gamma} \tag{10}$$

after substituting the pretension T_0 from equation (5).

Consequently the load assumed by each spoke is given by

$$P_i = T_0 \cos\gamma \cos\theta_i + \frac{A_S E_S}{L} a_S \cos^2\theta_i \tag{11}$$

with which the total load may be expressed as

$$P = \sum_{i=1}^{N} P_i = T_0 \cos\gamma \sum_{\theta=0}^{2\pi} \cos\theta_i + \frac{A_S E_S}{L} a_S \sum_{\theta=0}^{2\pi} \cos^2\theta_i \tag{12}$$

In this equation both sums are extended over all the spokes anchored to the rim and to the hub. So, if the number of spokes is *N*, there will be $N/2\pi$ spokes per angle unit, and the two sums can be calculated as follows:

$$\sum_{\theta=0}^{2\pi} \cos\theta_i = \int_0^{2\pi} \frac{N}{2\pi} \cos\theta d\theta = 0 \tag{13}$$

and $$\sum_{\theta=0}^{2\pi} \cos^2\theta_i = \int_0^{2\pi} \frac{N}{2\pi} \cos^2\theta d\theta = \frac{N}{2} \tag{14}$$

When these results are put into equation (12) it is obtained that

$$a_S = \frac{2L}{N A_S E_S} P \tag{15}$$

which is the spoke contribution to the radial distortion of the wheel.

Equation (15), shows effectively the proportionality between the distortion a_S and the load *P* and defines the stiffness of the wheel relative to the spokes as

$$K_S = \frac{N A_S E_S}{2L} \tag{16}$$

where the influence of the count of spokes and of their cross section and modulus of elasticity as well as the influence of the size of the wheel are quantified.

At this stage it may seem surprising that a_S does not depend on the pretension of the spokes. However, this is so due to the fact that all the spokes anchored to the hub, and pulling from it all around, experiment a tension equally increased by such a pretension T_0. This is shown if the result of equation (15) is taken into equation (10) yielding that

$$T_i = T_0 + \frac{2P}{N\cos\gamma}\cos\theta_i \qquad (17)$$

and explains why the magnitude of the pretension of the spokes does not influence the displacement a_S of the hub. However it is noted that the above calculations have been carried out on the assumption that the pretension of the spokes is such that they are kept in tension all the time.

Equation (17) also shows that the spokes in the upper half of the wheel, for which $-\pi/2 < \theta_i < \pi/2$ (see figure 3), are tensioned beyond the pretension when the load *P* is applied, whereas the spokes in the lower half of the wheel have their net tensile load reduced. The amount by which this occurs is due to spoke and rim relative stiffnesses and is studied in the next section.

Rim contribution

The wheel rim, when loaded, will experiment bending due to the reaction of the floor together with all the spoke tensions, which brings about another contribution a_R to the shortening of the distance between the hub and the contact point of the wheel with the floor. Evidently, this must be taken into account when calculating the stiffness of the wheel.

The approaching of the hub, upon which the load *P* is applied, to the lowest point of the rim, where the reaction *P* exerted by the floor acts, is due to the bending of the lower half part of the rim. This part of the rim is supported by the upper half and is subjected to the tension T_i of the spokes anchored to it and to the reaction of the floor *P*, as represented in figure 4.

Then the deflection a_R of the rim at the lowest point may be calculated by deriving the strain energy of the bent half part of the rim, by using the Castigliano´s theorem (Hearn, 1999, and Ryder, 1983).

Furthermore, accounting for symmetry reasons, the bending of the half rim is going to be studied by considering only a quarter of the wheel rim, as shown in figure 5, which supports half the reaction of the floor $P/2 = P_0$, the tension T_i of the spokes anchored to it and the hoop compression C_0 as well as the bending moment M_0 exerted by the rest of the wheel rim at the base.

Consequently, the upright deflection a_R at the base of the rim is obtained using Castigliano´s theorem from

$$a_R = \frac{\partial U}{\partial P_0} \tag{18}$$

where U is the strain energy of a quarter of the rim.

The strain energy stored in a quarter of the rim can be calculated from

$$U = \int \frac{M^2}{2E_R I_R} ds \tag{19}$$

where M is the bending moment in each section of the rim and I_R its moment of inertia, and the integration is computed between the limits of the quarter rim $\pi R/2$.

This calculation can readily be performed for a wheel having N number of spokes for the following conditions:

1.- the quarter of the circle is divided into $N/4$ segments (see figure 5) with spokes specified from the bottom (as 1, 2, ... j, etc) each having an arc length $2\pi R/N$ between consecutive spokes;

2.- in each segment the bending moment value M_j is taken to be the value at the central point.

Based on these assumptions the strain energy for the quarter of the rim can be expressed as

$$U = \frac{\pi R}{N E_R I_R} \sum_{j=1}^{N/4} M_j^2 \tag{20}$$

In this simplified expression, every moment M_j can be written as

$$M_j = M_j(P_0, C_0, M_0, T_i) \tag{21}$$

from figure 5, and thus equation (20) represents the total strain energy of the quarter of the rim as a function of the external loads P_0, C_0 and T_i and of the external couple M_0

$$U = U(P_0, C_0, T_i, M_0) \tag{22}$$

This now enables Castigliano´s theorem to be applied. Effectively, the upright deflection at the lowest point of the rim will be given by

$$a_R = \frac{\partial U}{\partial P_0} = \frac{\pi R}{N E_R I_R} \sum_{j=1}^{N/4} 2 M_j \frac{\partial M_j}{\partial P_0} \tag{23}$$

whereas the horizontal deflection and the angular rotation at this point, which must be null for symmetry reasons, will be given by

$$\frac{\partial U}{\partial C_0} = \frac{\pi R}{N E_R I_R} \sum_{j=1}^{N/4} 2 M_j \frac{\partial M_j}{\partial C_0} = 0 \tag{24}$$

and $$\frac{\partial U}{\partial M_0} = \frac{\pi R}{N E_R I_R} \sum_{j=1}^{N/4} 2 M_j \frac{\partial M_j}{\partial M_0} = 0 \tag{25}$$

Now, if each bending moment M_j is substituted from equation (21) and the tension of every spoke is taken as T_0, the last three equations (23), (24) and (25) allow to express the upright deflection at the lowest point of the rim as a function of the load P and of the pretension of the spokes T_0

$$a_R = a_R(P, T_0) \tag{26}$$

The deflection a_R determined in this way accounts for the bending of the rim due to both the load P and the spoke pretension T_0. It is obvious that only the term corresponding to the load P must be accounted for when determining the stiffness of the wheel rim, since the deflection caused by the spoke pretension will occur when the rim is being mounted as a joint structure and will have already been performed when the loading of the wheel takes place and the stiffness is being defined.

The whole procedure to calculate the rim contribution to the deformation of the wheel is going to be throughout illustrated in a particular case.

16 spoke wheel rim

In this section the rim contribution to the displacement of the hub towards the contact point of the rim with the floor, in a wheel with N=16 spokes, is worked out, as an example of how it can be carried out in any case and in order to analyse how it depends on the different variables which conform the rim.

The bending moments for the respective segments between consecutive spokes of the considered quarter wheel, as shown in figure 5, are as follows:

$$M_1 = P_0 R \sin\frac{\pi}{N} + T_1 R\cos\gamma\sin\frac{\pi}{N} - C_0 R\left(1-\cos\frac{\pi}{N}\right) - M_0 \qquad (27)$$

$$M_2 = P_0 R \sin\frac{3\pi}{N} + T_1 R\cos\gamma\sin\frac{3\pi}{N} + T_2 R\cos\gamma\sin\frac{\pi}{N} - C_0 R\left(1-\cos\frac{3\pi}{N}\right) - M_0$$

$$M_3 = P_0 R \sin\frac{5\pi}{N} + T_1 R\cos\gamma\sin\frac{5\pi}{N} + T_2 R\cos\gamma\sin\frac{3\pi}{N} + T_3 R\cos\gamma\sin\frac{\pi}{N} + \\ - C_0 R\left(1-\cos\frac{5\pi}{N}\right) - M_0$$

$$M_4 = P_0 R \sin\frac{7\pi}{N} + T_1 R\cos\gamma\sin\frac{7\pi}{N} + T_2 R\cos\gamma\sin\frac{5\pi}{N} + T_3 R\cos\gamma\sin\frac{3\pi}{N} + \\ + T_4 R\cos\gamma\sin\frac{\pi}{N} - C_0 R\left(1-\cos\frac{7\pi}{N}\right) - M_0$$

Substituting these values into equation (23) gives the deflection a_R

$$a_R = \frac{2\pi R^2}{N E_R I_R}\left[M_1\sin\frac{\pi}{N} + M_2\sin\frac{3\pi}{N} + M_3\sin\frac{5\pi}{N} + M_4\sin\frac{7\pi}{N}\right] \qquad (28)$$

whereas equations (24) and (25) reduce to

$$M_1 \cos\frac{\pi}{N} + M_2 \cos\frac{3\pi}{N} + M_3 \cos\frac{5\pi}{N} + M_4 \cos\frac{7\pi}{N} = 0 \qquad (29)$$

and $$M_1 + M_2 + M_3 + M_4 = 0 \qquad (30)$$

Now, if $N = 16$ and $T_i = T_0$, equations (27) are simplified as follows :

$$\begin{aligned}
M_1 &= 0.19509 P_0 R + 0.19509 T_0 R \cos\gamma - 0.01921 C_0 R - M_0 \\
M_2 &= 0.55557 P_0 R + 0.75066 T_0 R \cos\gamma - 0.16853 C_0 R - M_0 \\
M_3 &= 0.83147 P_0 R + 1.58213 T_0 R \cos\gamma - 0.44443 C_0 R - M_0 \\
M_4 &= 0.98079 P_0 R + 2.56292 T_0 R \cos\gamma - 0.80491 C_0 R - M_0
\end{aligned} \qquad (31)$$

Then equations (29) and (30) represent a simple system from which the hoop compression C_0 and the bending moment M_0 are directly obtained in terms of the load P_0 and of the pretension of the spokes T_0 as follows:

$$C_0 = 0.93771 P_0 + 2.98247 T_0 \cos\gamma \qquad (32)$$

and $$M_0 = 0.30384 P_0 R + 0.20118 T_0 R \cos\gamma \qquad (33)$$

Afterwards, taking values (32) and (33) to expressions (31) gives

$$\begin{aligned}
M_1 &= -0.12676 P_0 R - 0.06338 T_0 R \cos\gamma \\
M_2 &= 0.09370 P_0 R + 0.04684 T_0 R \cos\gamma
\end{aligned} \qquad (34)$$

$$M_3 = 0.11088 P_0 R + 0.05545 T_0 R \cos\gamma$$

$$M_4 = -0.07782 P_0 R - 0.03888 T_0 R \cos\gamma$$

Finally, by substituting values (34) in expression (28) and $P_0 = P/2$, it is obtained that

$$a_R = \frac{\pi R^3}{8 E_R I_R}\left[0.02160 P + 0.02163 T_0 \cos\gamma\right] \tag{35}$$

where the two terms of the deflection of the rim appear. As it was said before only the term due to the load *P* is accountable when determining the stiffness of the wheel.

Consequently, the stiffness relative to the rim, when it has 16 spokes is given by

$$K_R = \frac{8 E_R I_R}{0.06786 R^3} \tag{36}$$

Radial stiffness of the wheel

Now the total displacement *a* of the hub caused by the in-plane load *P* may be expressed by adding a_S and the corresponding term of a_R

$$a = a_S + a_R = \frac{P}{K_S} + \frac{P}{K_R} = P \frac{K_S + K_R}{K_S K_R} \tag{37}$$

in which the stiffness of the wheel (as a structure comprising rim and spokes) is computed as

$$K_W = \frac{1}{\dfrac{1}{K_S} + \dfrac{1}{K_R}} = \frac{K_S K_R}{K_S + K_R} = \frac{K_R}{1 + \dfrac{K_R}{K_S}} \tag{38}$$

In the case of a wheel having 16 spokes, substituting equations (16) and (36) into equation (38) yields the result

$$K_W = \frac{8}{\dfrac{L}{A_S E_S} + \dfrac{0.06786 R^3}{E_R I_R}} \tag{39}$$

This equation quantifies the contributions of both the spokes and the rim, in terms of their geometry and materials, to the stiffness of the wheel.

If, for example, the spoke stiffness is significantly greater than that of the rim, such that K_R / K_S is small then the overall wheel stiffness is dominated by the less stiff rim. This can be seen from equation (38) by eliminating the K_R/K_S ratio and leaving K_W ---> K_R.

Pretension of the spokes

When studying the spoke contribution to the stiffness of the wheel, the length of every spoke, after the wheel is loaded, was named L_i and expressed in equation (8) relative to $R/\cos\gamma$, which is the length the spoke had reached after being anchored and fixed between the hub and the rim, with a pretension T_0.

The spokes situated below the hub, for which $|\theta_i| > \pi/2$ (see figure 3) and $L_i < R/\cos\gamma$, experiment a shortening when the wheel is loaded and so a reduction of their tension, from the pretension T_0 to $T_i < T_0$, according to equation (17). The biggest shortening corresponds to the spoke at the lowest position and is worth, from equation (15),

$$\frac{a_S}{\cos\gamma} = \frac{L}{A_S E_S}\frac{P}{8} = \frac{R}{A_S E_S}\frac{P}{8\cos\gamma} \tag{40}$$

when the number of spokes is 16.

Also, the bending of the low half part of the rim due to the reaction of the floor and to the tension of the spokes anchored to it causes an additional shortening of these spokes and the subsequent reduction of their tension. This shortening is maximum at the lowest position where it is worth [(see equation (35)]

$$\frac{a_R}{\cos\gamma} = \frac{0.06786R^3}{E_R I_R}\frac{P}{8\cos\gamma} \tag{41}$$

again, when the number of spokes is 16.

It is important to recognise that the developed theory assumes that spokes remain in tension at all times. This is because a spoke is incapable of supporting a significant compressive load due to its high slenderness (that is length to diameter ratio) thus making it prone to buckle. This means that it would have been incorrect to have assumed that a spoke could support any appreciable compressive load and so means that spokes must remain in tension at all times. This leads to the requirement that the axial strain extension resulting from pre-tensioning needs to exceed the greatest shortening that a spoke will experience – which is when passing through their lowest position. Therefore, based on the 16 spoke wheel examples, the following condition must be met [see equations (40) and (41)]:

$$\frac{T_0 L}{A_S E_S} \geq \left[\frac{R}{A_S E_S} + \frac{0.06786R^3}{E_R I_R}\right]\frac{P}{8\cos\gamma} \tag{42}$$

from which the minimum pretension the spokes need is deduced

$$T_0 \geq \left[1 + 0.06786\frac{E_S}{E_R}\frac{A_S R^2}{I_R}\right]\frac{P}{8} \tag{43}$$

This will ensure that all the spokes are always tensioned, whatever their position within the wheel may be, which is the most convenient condition for the stability and strength of the wheel.

Maximum tension of the spokes

Considering that the spokes in the upper half part of the rim are not as highly influenced by the bending of the lower half part, equation (17) is valid for them and may be used to calculate their tension, which depends on their position within the wheel. Evidently, the maximum tension occurs in the highest spoke, for which $\theta_i = 0$, and is worth

$$T_{\max} = T_0 + \frac{2P}{N\cos\gamma} = T_0 + \frac{P}{8\cos\gamma} \tag{44}$$

if the number of spokes is 16. Then the necessity for the tension never being negative makes condition (43) unavoidable and therefore

$$T_{\max} \geq \left[1 + 0.06786\frac{E_S}{E_R}\frac{A_S R^2}{I_R} + \frac{1}{\cos\gamma}\right]\frac{P}{8} \tag{45}$$

This result does not give the tension of the highest strained spoke but its minimum value as long as the pretension ensures that no spoke will be in compression. In fact, the maximum tension the spokes will be subjected to depends very much on the pretension T_0 they are provided with under condition (43).

Numerical applications

It is interesting to apply the theoretical results so far developed to some specific cases so that the significance of individual contributions can be seen. The examples considered are taken from Chandler (2002) and correspond to three commercially available wheel models, namely the Mavic Open Pro, the Shimano WH-6500 and the Rolf Vector Pro as shown in figure 6. Their structural properties are given in table 1.

Although the actual wheels are fitted with different numbers of spokes and with different mounting arrangements, in this work the three rims were assumed to have 16 spokes with a length equal to the radius of the rim, $L = R = 350\,\text{mm}$ ($\cos\gamma \approx 1$), according to the idealised model of figure 1. In this way the results will enable direct comparisons to be made. Also, in the three cases the rim material was aluminium alloy ($E_R = 70\,\text{GN/m}^2$), whereas the spokes are from steel

($E_S = 200\,\mathrm{GN/m^2}$). The diameter of the spokes in all cases is taken as 2 mm ($A_S = \pi\,\mathrm{mm^2}$).

Using this data the radial stiffnesses for spokes and rims were obtained using equations (16), (36) and (38) and results are given in table 2. These results emphasize the significant differences in the magnitudes of the spoke and the rim stiffnesses and thus their contributions to the wheel as a structural assembly of the two elements.

Table 3 presents the minimum pretension loads of the spokes for the three cases, as calculated from equation (43), alongside with the maximum tension the spokes will be subjected to, as calculated from equation (45). Both the pretension and the maximum tension are given relative to the load applied on the hub.

General discussion

Although this analytical study of a spoked bicycle wheel is not exhaustive, the analysis has established some important findings regarding the contributions made by the rim and the spokes to the structural behaviour of a wheel.

In this work the effect of axial load distribution in the spokes and the bending of the rim are combined to establish the radial stiffness of the wheel. It is found that the results of this analysis are of the same order as those published by Chandler (2002) by Hopkins and Principle (1990) and by Grignon (2002), which suggests that the values are of the right order of magnitude.

The study has found that, in general, the system of spokes is much stiffer than that of the rim and so the rim stiffness largely dictates the wheel radial stiffness. This is particularly evident when studying equation (38) which shows how the lower value of rim stiffness dominates the wheel stiffness when the system of spokes have a stiffness an order of magnitude greater. It is interesting to observe that modern optimal wheel design is moving towards wheels having much stiffer rim cross sections with fewer spokes. For the radial stiffness of such wheels, which have a wider spoke spacing distance, to compare with traditional multi-spoke wheels, it is only possible when the cross section is much stiffer in bending.

In table 2, and with the help of equation (38), it can be deduced that significantly changing the stiffness of the spokes will have little effect on the general stiffness of the wheel. Consequently, reducing the number of spokes is considered desirable as long as the wheel remains stable and so rim bending rigidity becomes of critical importance. It is interesting to observe how the Rolf Vector Pro rim despite having a smaller cross section with respect to the Shimano WH-6500 rim, and thus less weight and less material usage, has an improved stiffness of the wheel due to its increased moment of inertia (see tables 1 and 2).

In addition, if the rim has a greater bending stiffness, spoke pretension need not be so great thus permitting a lower maximum tension. This is because the stiffer the rim, the lower the bending deflection in the bottom half of the rim and the smaller the spoke contraction necessary with a lower pretension [see equation (43)]. Consequently with a lower pretension load when the wheel rotates half a turn and the lower spokes come to the upper half part of the wheel they will not be placed under such high tension [see equation (45)].

However, the rim moment of inertia cannot be increased indefinitely as there exists a limit to the number of pre-tensioned spokes possible and so a compromise is necessary for an optimum design to be achieved.

Conclusions

This analytical study has helped quantify how the spokes and the rim interact with each other and work together when assembled to form a wheel and subjected to a radial load. The work has taken into account both the bending stiffness of the rim section and the axial tensile stiffness and preloading of the set of spokes in determining a wheel's radial stiffness.

It was deduced that achieving an optimum wheel design, in terms of wheel radial stiffness, necessitates making a compromise between maximising rim cross-section bending stiffness (which permits greater distance between successive spokes) and having fewer spokes. Because there are many spoke and rim defining parameters, many having conflicting effects, achieving an optimum wheel is a complex process and so analytical equations have been developed to help simplify the design optimisation process.

Pretension of the spokes is an essential part of achieving an efficient wheel and the developed theory has enabled the minimum possible magnitude to be determined so that spokes always remain under tension throughout the wheel rotation load cycle.

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Figure captions

Figure 1.- Idealised spoked wheel: the spokes are anchored to the rim and to the hub. The load is applied on the hub.

Figure 2.- Half wheel rim supporting the tensions of the spokes fixed to it and the hoop compression.

Figure 3.- When the external load P is applied on the hub it is displaced a_S, and so a spoke, whose pretension was T_1, is deflected and enlarged $a_S \cos\theta_i$, while its tension becomes T_i.

Figure 4.- The wheel rim is bent by the reaction P of the floor. The deflection at the contact point a_R is the rim contribution to the radial distortion of the wheel.

Figure 5.- Quarter of the rim supported by the rest of the rim at section A and subjected to the tension of the spokes anchored to it and to half the reaction of the floor. The action of the rest of the rim at the bottom (C_0 and M_0) is considered also as an external load.

Figure 6.- Different rim cross sections.

Rim	Mavic Open Pro	Shimano WH-6500	Rolf Vector Pro
Cross-Section Area A_R (mm^2)	81.9	96.6	95.1
Moment of Inertia I_R (mm^4)	4090	7670	8930

Table 1: Rim structural properties.

Rim	Mavic Open Pro	Shimano WH-6500	Rolf Vector Pro
Stiffness due to spokes K_S (MN/m)	14.362		
Stiffness due to rim K_R (MN/m)	0.787	1.476	1.719
Resultant Radial Stiffness K (MN/m)	0.746	1.338	1.535

Table 2: Radial stiffness of wheels.

Rim	Mavic Open Pro	Shimano WH-6500	Rolf Vector Pro
Pretension $T_0 \geq$	2.41 *P*	1.34 *P*	1.17 *P*
Maximum Tension $T_{\max} \geq$	2.53 *P*	1.47 *P*	1.29 *P*

Table 3: Spoke pretension and maximum tension, *P* being the spoke total load.