On the relative energies of the Rivlin and Cavitation instabilities for Compressible Materials

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June 29th, 2011

Abstract

In 1948, R.S. Rivlin showed that if a cube of incompressible neo-Hookean material is subjected to a sufficiently large uniform, normal, dead-load on its boundary, then an asymmetric deformation is the minimiser of the energy in the class of homogeneous deformations. J.M. Ball showed in 1982 that, for classes of compressible elastic materials, if a ball of the material is subjected to a sufficiently large uniform radial dead-load, then a deformation forming a cavity is the minimiser of the energy in the class of radial deformations. In this paper we treat *compressible* hyperelastic materials and show that under such dead-loading, if a local minimiser of the radial energy forms a cavity, then there necessarily exists an asymmetric homogeneous deformation with less energy. Our approach extends and generalises previous results of Abeyaratne and Hou for the incompressible case.

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1 Introduction

Consider a ball of compressible, hyperelastic, isotropic material occupying the region $B := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\}$ in its reference configuration. We consider deformations $\mathbf{u} : B \to \mathbb{R}^n, \mathbf{u} \in W^{1,q}(B;\mathbb{R}^n)$ for $n = 2,3$ and for $q < n$. The bulk energy stored in the deformed body under such a deformation is given by

$$E(\mathbf{u}) = \int_B W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1.1}$$

where $W : M_+^{n\times n} \to \mathbb{R}$ is the stored energy function and $M_+^{n\times n}$ denotes the real $n \times n$ matrices with positive determinant. The equilibrium equations for (1.1) are the Euler -Lagrange equations given by

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla \mathbf{u})\right] = 0. \tag{1.2}$$

It is well known that W is isotropic if and only if there exists a symmetric function

$$\Phi : \mathbb{R}^n_{++} \to \mathbb{R}, \qquad \mathbb{R}^n_{++} = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \ : \ x_i > 0\},$$

such that

$$W(\mathbf{F}) = \Phi(v_1, v_2, ..., v_n) \qquad \forall\, \mathbf{F} \in M_+^{n\times n}, \tag{1.3}$$

where the v_i are the singular values of $\mathbf{F}$, which are also called the principal stretches. We write $\Phi_i \equiv \frac{\mathrm{d}\Phi}{\mathrm{d}v_i}$, etc. We assume that $\Phi_i(1,1,...) = 0$ so that the undeformed configuration is a natural state.

The Piola-Kirchhoff stress tensor $\mathbf{S}$ is given by

$$\mathbf{S} = \frac{\partial \Phi}{\partial \mathbf{F}}. \tag{1.4}$$

In the dead-load traction boundary value problem we specify

$$\mathbf{SN} = \mathbf{t} \quad \text{on} \quad \partial B, \tag{1.5}$$

where $\mathbf{t}$ is given, $\mathbf{N}$ is the unit outward normal on ∂B and $P > 0$ is the applied normal stress. The associated energy functional is then given by

$$E_P(\mathbf{u}) = \int_B W(\nabla \mathbf{u})\mathrm{d}\mathbf{x} - \int_{\partial B} \mathbf{t}.\mathbf{u}\ \mathrm{d}S \tag{1.6}$$

and (1.5) is then the natural boundary condition associated with (1.6). In this paper we consider the case in which the ball is subjected to a uniform radial dead-loading on the boundary so that

$$\mathbf{SN} = P\mathbf{N} \quad \text{on} \quad \partial B, \tag{1.7}$$

where $\mathbf{N}$ is the unit outward normal on ∂B and $P > 0$ is the applied normal stress. Hence, the associated energy functional takes the form

$$E_P(\mathbf{u}) = \int_B W(\nabla \mathbf{u}) \mathrm{d}\mathbf{x} - \int_{\partial B} P\mathbf{u}.\mathbf{N} \; \mathrm{d}S. \tag{1.8}$$

1.1 Rivlin Instability

Homogeneous deformations $\mathbf{u}^h$ are deformations of the form $\mathbf{u}^h(\mathbf{x}) \equiv A\mathbf{x} + \mathbf{c}$ and these always satisfy (1.2). The total energy (1.8) of a homogeneous deformation of B is then given by[1]

$$E_a(\mathbf{u}^h) = \frac{\omega_n}{n} \left[W(A) - P \, tr(A) \right], \tag{1.9}$$

and minimisers in the class of homogeneous deformations will satisfy the natural boundary condition (1.7).

Consider the class of purely homogeneous deformations which take the form

$$\mathbf{u}(\mathbf{x}) = (v_1 x_1, ..., v_n x_n) \qquad\qquad v_1, ..., v_n > 0, \tag{1.10}$$

for $\mathbf{x} = (x_1, ..., x_n) \in B$. The principal stretches associated with (1.10) are clearly $v_1, ..., v_n$ and the total energy associated with any such deformation, for the dead-load traction problem with radial dead-load P, is then given by

$$E_a(v_1, ..., v_n) := E_P(\mathbf{u}) = \frac{\omega_n}{n} \left[\Phi(v_1, ..., v_n) - P(v_1 + ... + v_n) \right]. \tag{1.11}$$

The Euler-Lagrange equations[2] for (1.11) are

$$\Phi_i(v_1, ..., v_n) - P = 0, \qquad \text{for } i = 1, ..., n. \tag{1.12}$$

Rivlin [1], [2] studied bifurcation of solutions to the dead-load traction problem for an *incompressible neo-Hookean* material in the class of purely homogeneous deformations. He showed that, for small P, the only homogeneous solution was $\mathbf{u}^h(\mathbf{x}) \equiv \mathbf{x}$,

[1] Where ω_n denotes the area of the unit sphere in $\mathbb{R}^n$.

[2] Clearly a minimiser exists for continuous energy functions (1.3) under mild growth assumptions such as $\frac{W(\mathbf{F})}{|\mathbf{F}|} \to \infty$ as $|\mathbf{F}| \to \infty$.

but that, for larger values of P, there was a bifurcation into an asymmetric homogeneous state of the form $\mathbf{u}(\mathbf{x}) \equiv \mathrm{diag}(\mu, \nu, \nu)\mathbf{x}$ (here $n = 3$) with $\mu\nu^2 = 1,\ \mu \neq \nu$. This result can be obtained by analysing the bifurcation of solutions v_1, v_2, v_3 to the algebraic equations

$$\Phi_{,i}(v_1, v_2, v_3) - p(v_i^{-1}) = P, \quad i = 1, 2, 3, \tag{1.13}$$

from the trivial solution $v_1 = v_2 = v_3 = 1$, under the constraint $v_1 v_2 v_3 = 1$ (where p is the hydrostatic pressure).

Although Rivlin's results were derived ostensibly for deformations of a cube, they apply to any reference shape of body. Rivlin's analysis was subsequently extended and generalised to separable stored energy functions in an interesting paper by Ball and Schaeffer [3] (and they note that their methods extend to more general forms of the energy function). In particular, they studied Mooney-Rivlin materials, which have the form

$$\Phi(v_1, v_2, v_3) = \frac{\mu}{2}(v_1^2 + v_2^2 + v_3^2 - 3) + \frac{\mu}{2k}(v_1^{-2} + v_2^{-2} + v_3^{-2} - 3),$$

where $\mu, k > 0$, and they were able to give a comprehensive analysis of the existence and stability properties of the asymmetric bifurcating homogeneous solutions in this case. They found that for large dead-loads the trivial solution became unstable and that there was a bifurcation into branches of "rod-like" and "plate-like" solutions. For $k \leq 3$ the plate-like solutions are always unstable and the rod-like ones neutrally stable[3]. For finite $k > 3$, the plate-like solutions are initially neutrally stable (with the rod -like solutions being unstable) and as the dead-load increases they found a secondary bifurcation into a neutrally stable solution with three unequal stretches which joined the plate-like branch to the rod-like branch (after this point the rod-like solutions became neutrally stable). Ball and Schaeffer also adapted an argument of Rivlin [5] to argue that the bifurcation picture would remain qualitatively the same in the case of a slightly compressible Mooney-Rivlin stored energy function.

It is not our intention in this paper to give comprehensive references to the large body of existing work on the Rivlin cube problem however, for illustration, we mention a number of such contributions and refer the interested reader to these and the references therein. Rivlin and Beatty [6] studied the stability of pure homogeneous deformations of a compressible cube subjected to a dead-load. They found necessary and sufficient conditions for stability of purely homogeneous deformations. They found that for neo-Hookean materials, in the limiting case of incompressibility, these

[3] In fact, they showed that any non-trivial equilibrium solution is at most neutrally stable (see also previous work in [4]).

conditions were the same as those found by Rivlin in [1], [2]. Tarantino [7] has also studied the compressible problem, including a numerical study for neo-Hookean and Mooney-Rivlin materials. His numerical results corroborated the analytic results of Ball and Schaeffer [3] for certain compressible versions of the Mooney-Rivlin stored energy. Extensions to the case of anisotropic materials are contained in the work of [8]. Other interesting contributions include the work of Sawyers [9], Ogden (see, e.g., [10]), Chen and MacSithigh (see, e.g., [11], [12],[13]).

There has also been much work on the related phenomena of bifurcation into asymmetric homogeneous deformations of a square membrane of incompressible elastic material under uniform planar dead-loading. Steigmann [14] found necessary and sufficient conditions for stability in isotropic materials. Haughton [15] has studied bifurcation and stability of solutions of a biaxially loaded cube made of a compressible material. He found that, for sufficiently large loads, the trivial solution loses its stability and bifurcates into a neutrally stable asymmetric homogeneous solution. Other contributions include work of Kearsley [16], Macsithigh [17] and Batra et al [18].

1.2 Cavitation Instability

Ball [19] studied energy minimisers for the dead-load traction problem in the class of radial deformations

$$\mathbf{u}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x} \quad \text{for } \mathbf{x} \in B,\ R = |\mathbf{x}|, \tag{1.14}$$

of a ball. In the case of radial deformations, the principal stretches are $v_1 = v_r$ and $v_2 = ... = v_n = v_\theta$ where

$$v_r = r'(R), \qquad v_\theta = r(R)/R \qquad \text{for } 0 < R < 1. \tag{1.15}$$

The the energy functional (1.8) takes the form

$$E_P(\mathbf{u}) = \omega_n I_P(r) = \omega_n \left(\int_0^1 R^{n-1}\Phi\left(r', \frac{r}{R}, .., \frac{r}{R}\right)\, dR - Pr(1) \right), \tag{1.16}$$

and the corresponding equilibrium equations (1.2) reduce to the single ordinary differential equation

$$\frac{d}{dR}\left[R^{n-1}\Phi_{,1}\left(r', \frac{r}{R}, .., \frac{r}{R}\right)\right] = (n-1)R^{n-2}\Phi_{,2}\left(r', \frac{r}{R}, .., \frac{r}{R}\right), \tag{1.17}$$

called the radial equilibrium equation. The dead-load boundary condition (1.7) on the outer boundary of the ball takes the form

$$\Phi_{,1}\left(r'(1),\ r(1),..,r(1)\right) = P. \tag{1.18}$$

Ball showed that for sufficiently large P any energy minimiser of (1.16) must satisfy $r(0) > 0$ (so that the deformed ball contains a hole of this radius) and the natural boundary condition

$$\lim_{R\to\infty} T(r(R)) = 0, \tag{1.19}$$

where T is that radial component of the Cauchy stress given by

$$T(r(R)) = \left(\frac{R}{r(R)}\right)^{n-1} \Phi_{,1}\left(r'(R), \frac{r(R)}{R}, .., \frac{r(R)}{R}\right). \tag{1.20}$$

For further references on this radial problem see [20] and [21].

1.3 Relative occurrence of the two instabilities

For the incompressible case, Ball comments in [19] that the critical value of the dead-load at which bifurcation to the Rivlin instability occurs may be greater or less than the critical value at which bifurcation into the cavitation instability occurs, depending on the material parameters.

In an interesting paper Abeyaratne and Hou [22] show that for *incompressible* materials, the cavitating radial energy minimiser of Ball is never a *global* minimiser in any class of deformations that includes the radial deformations and the asymmetric homogeneous deformations studied by Rivlin. In particular, they showed that, for any given value of the applied stress $P > 0$, if there exists a stable cavitation deformation, then necessarily there exists a stable asymmetric homogeneous deformation with less energy for the same value of P.

Hou [23] has taken this comparison further for neo-Hookean materials. He defines a class of semi-inverse deformations that includes both radial cavitating deformations and asymmetric homogeneous deformations as well as some axisymmetric deformations. In particular, Hou was able to show that the radial cavitated deformations are unstable in this larger class of deformations for moderate P but that they become stable for sufficiently large P.

In the current paper we extend and generalise the results of Abeyaratne and Hou [22] to the case of *compressible* materials. A major difficulty in effecting such an extension is that, in the incompressible case, the radial cavitation solution is explicitly known from the kinematic constraint of incompressibility, whereas, in the

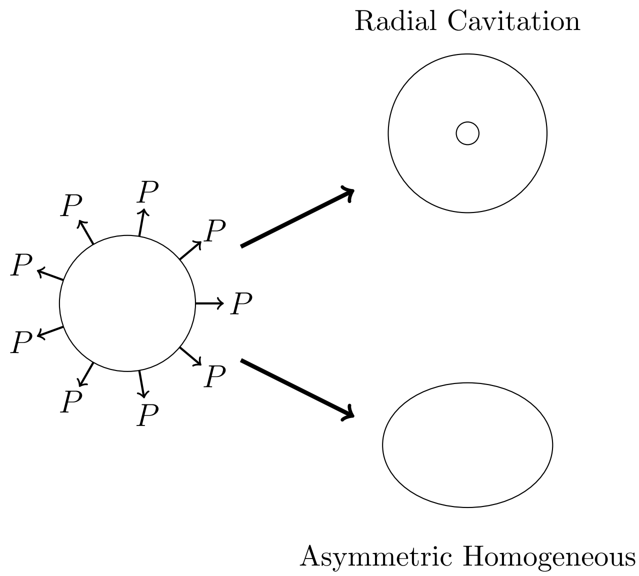

Figure 1: Cavitation instability and the Rivlin instability

compressible case, it is only implicitly defined as a solution of the radial equilibrium equation (1.17). Though our methods in this compressible case are therefore very different, we would like to acknowledge the many insights we have gained from the work of Abeyaratne and Hou for the incompressible problem.

The main steps in the proof of our main result are:

(i) The total energy (1.8) of a radial cavitation solution $\mathbf{u}$ (i.e. of the form (2.3)) to the dead-load traction problem can be expressed in terms of the values of $\mathbf{u}, \nabla\mathbf{u}$ on the boundary $\partial\Omega$ (this follows from a divergence identity due originally to A.E. Green)

(ii) This total energy can be shown to be equal to the total energy of a related asymmetric homogeneous deformation,

(iii) This related (asymmetric) homogeneous deformation is not in general a global minimiser of the total energy in the class of all homogeneous deformations (see Theorem 4.2). Hence the radial cavitation solution cannot be a global energy minimiser.

Our approach yields the results of [22] as a special case (see concluding remarks in section 5).

2 The energy of a smooth solution of the dead-load traction problem.

In this section we calculate the total energy (1.8) of a smooth (not necessarily symmetric) solution of the dead-load traction boundary value problem: i.e., a solution of (1.2) on some domain Ω that satisfies the boundary condition (1.7) on $\partial\Omega$. To do this we use the following divergence identity for smooth solutions of (1.2) which is originally due to Green:

$$\mathrm{Div}\left[W(\nabla\mathbf{u})\mathbf{x} + \left(\frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u})\right)^T(\mathbf{u} - \nabla\mathbf{u}\mathbf{x})\right] = nW(\nabla\mathbf{u}). \tag{2.1}$$

If we apply this to an equilibrium solution $\mathbf{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ on the domain Ω with an imposed dead-load $\mathbf{t} = P\mathbf{N}$, $P > 0$, on its boundary $\partial\Omega$ we find by (1.8) that

$$\begin{aligned} E_P(\mathbf{u}) &= \frac{1}{n}\int_{\partial\Omega} W(\nabla\mathbf{u})\mathbf{x}.\mathbf{N}\,\mathrm{d}S - \frac{n-1}{n}\int_{\partial\Omega} t^i u^i\,\mathrm{d}S - \frac{1}{n}\int_{\partial\Omega} u^i_{,k}x^k t^i\,\mathrm{d}S \\ &= \frac{1}{n}\int_{\partial\Omega} W(\nabla\mathbf{u})\mathbf{x}.\mathbf{N}\,\mathrm{d}S - P\int_{\partial\Omega}\left(\frac{(n-1)}{n}\mathbf{u} + \frac{1}{n}(\nabla\mathbf{u}\,\mathbf{x})\right).\mathbf{N}\,\mathrm{d}S. \end{aligned} \tag{2.2}$$

Now suppose that $\Omega = B$ and that $\mathbf{u} \in C^2(\bar{B}\backslash\{\mathbf{0}\})$ is a cavitating radial equilibrium solution, then

$$\mathbf{u}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x} \quad \text{for } \mathbf{x} \in B,\ R = |\mathbf{x}|, \tag{2.3}$$

where $r(0) > 0$ and (1.19) is satisfied. In this case, the singular values of $\nabla\mathbf{u}$ are constant on ∂B and are given by $v_1 = r'(1), v_2 = r(1), ..., v_n = r(1)$ on ∂B.[4] Hence, the first part of the energy integral in (2.2) is equal to

$$\frac{1}{n}\int_{\partial B} W(\nabla\mathbf{u})\mathbf{x}.\mathbf{N}\,\mathrm{d}S = \frac{1}{n}\int_{\partial B}\Phi(\alpha,\beta,...,\beta)\mathbf{x}.\mathbf{N}\,\mathrm{d}S = \int_B W(\nabla\mathbf{u}^h)\,dx, \tag{2.4}$$

where $\mathbf{u}^h$ the corresponding homogeneous deformation

$$\mathbf{u}^h(\mathbf{x}) \equiv (\alpha x_1, \beta x_2, ..., \beta x_n) \tag{2.5}$$

[4] Our derivation of (2.2) assumed smoothness of the solution throughout B. However, the same result can be obtained by first excluding a small neighbourhood $B_\epsilon(\mathbf{0})$ around the origin, then applying the above argument to $\Omega_\epsilon = B\backslash B_\epsilon(\mathbf{0})$, and finally letting $\epsilon \to 0$. In this process, it can be shown that the contribution from the boundary integral on ∂B_ϵ converges to zero as $\epsilon \to 0$ by (1.19)- see [24, Section 3].

with

$$\alpha = r'(1), \beta = r(1). \tag{2.6}$$

On noting that $\mathbf{N} = \mathbf{x}$ on ∂B, it follows that the second integral term in (2.2) is then given by

$$\begin{aligned}
\int_{\partial\Omega} \left(\frac{(n-1)}{n}\mathbf{u} + \frac{1}{n}(\nabla \mathbf{u}\,\mathbf{x}) \right) .\mathbf{N}\,\mathrm{d}S &= \int_{\partial\Omega} \left(\frac{(n-1)}{n} r(1)\mathbf{x} + \frac{1}{n}(r'(1)\mathbf{x}) \right) .\mathbf{N}\,\mathrm{d}S \\
&= \int_{\partial\Omega} \left(\frac{(n-1)}{n}\beta + \frac{1}{n}(\alpha) \right) \mathrm{d}S \\
&= \frac{1}{n}\int_{\partial\Omega} \left(\beta \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \alpha \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) .\mathbf{N}\,\mathrm{d}S \\
&= \frac{1}{n}\int_{\partial\Omega} \mathbf{u}^h.\mathbf{n}\,\mathrm{d}S
\end{aligned} \tag{2.7}$$

and hence by (2.4), (2.7) it follows that

$$E_P(\mathbf{u}^h) = E_P(\mathbf{u}), \tag{2.8}$$

so that the energy of the radial cavitation solution equals the energy of the corresponding homogeneous deformation (2.5).

Remark 2.1. *The last result* (2.8) *could be obtained more directly using the radial version of the conservation law* (2.1) *(see, e.g., [19, expression (6.12)]), however, our intention in the above derivation is to highlight the general n-dimensional structure of the underlying problem.*

3 The energies associated with the cavitation and Rivlin instabilities

Having obtained the above result, our approach in the following sections is to now show that the homogeneous deformation (2.5) is not a global minimiser in the sub class of asymmetric deformations of the general form

$$\mathbf{u}^h(\mathbf{x}) \equiv (\gamma x_1, \delta x_2, ..., \delta x_n),\ \gamma, \delta > 0. \tag{3.1}$$

To this end we first recall a basic property of the radial cavitation solutions.

3.1 Energy of the radial cavitation deformations

Proposition 3.1. *Given any cavitation solution* $r \in C^2((0,1))$ *(i.e. a solution of* (1.17) *satisfying* $r(0) > 0$ *and the natural boundary condition* (1.19)*), this can be extended to a solution* $r_0 \in C^2((0,\infty))$ *satisfying. Given any cavitation solution* r *of the dead-load traction problem with dead-load* P*, there exists* $\alpha > 0$ *such that* $r(R) \equiv \alpha r_0\left(\frac{R}{\alpha}\right)$*. Moreover, by rescaling* r_0 *if necessary, we may assume without loss of generality that* $r_0(0) = 1$ *so that* α *corresponds to the radius of the cavity formed by the deformation.*

For each $\mu \in (\lambda_{\text{crit}}, \infty)$ we define $\alpha(\mu)$ to be the unique positive number satisfying

$$\mu = \alpha(\mu) r_0\left(\frac{1}{\alpha(\mu)}\right) \tag{3.2}$$

so that $\alpha(\mu)$ corresponds to the radius of the cavity formed by a cavitating deformation that is equal to μ on the outer boundary of the ball.

We refer to [25] for a variety of hypotheses on the stored energy function under which the last proposition holds. A simple class of such energy functions is given by the class of Ogden materials in the following example.

Example 3.2. *The following class of energy functions are such that Proposition 3.1 holds:*

$$\Phi(v_1, ..., v_n) = \sum_{j=1}^{M} \eta_j \left(\sum_{i=1}^{n} v_i^{\alpha_j} - n\right) + \sum_{j=1}^{N} \nu_j \left(\sum_{i=1}^{n} v_i^{\beta_j} - n\right) + h(v_1 ... v_n),$$

where $M, N \geq 0$*,* $\eta_j > 0$ *for* $1 \leq j \leq M$*,* $\nu_j > 0$ *for* $1 \leq j \leq N$*,* $n > \alpha_1 \geq ... \geq \alpha_M \geq 1$*,* $n/(n-1) > \beta_1 \geq ... \geq \beta_N \geq 0$ *and* $h : (0,\infty) \to \mathbb{R}$ *is* C^2*, strictly convex and satisfies*

$$\lim_{\delta \to 0^+} h(\delta) = \lim_{\delta \to \infty} \frac{h(\delta)}{\delta} = \infty \tag{3.3}$$

and

$$|h'(d)d| \leq const.\left(1 + h(d)\right), \text{ for all } d > 0. \tag{3.4}$$

We can use (3.2) to parametrise the energy of a cavitation solution using μ and we correspondingly define

$$\hat{E}_P(\mu) = \omega_n I_P\left(r_\mu\right), \quad r_\mu(R) := \alpha(\mu) r_0\left(\frac{R}{\alpha(\mu)}\right). \tag{3.5}$$

On making the change of variables $\tilde{R} = R/\alpha(\mu)$ in (3.5) we obtain

$$\widehat{E}_P(\mu) = \omega_n \left[\int_0^{1/\alpha(\mu)} (\alpha(\mu))^n \tilde{R}^{n-1} \Phi(r_0'(\tilde{R}), r_0(\tilde{R})/\tilde{R}, ...) \mathrm{d}\tilde{R} - P\mu \right], \tag{3.6}$$

where $\widehat{E}_P(\mu) = E_P(r_\mu)$.

We will now perform a second change of variables so that the new variable of integration is the tangential principal stretch through the body, $\nu = \frac{r_0(\tilde{R})}{\tilde{R}}$. We note here that by definition $\nu = \alpha(\nu) r_0(1/\alpha(\nu))$, which implies that $\tilde{R} = 1/\alpha(\nu)$. This then yields

$$\widehat{E}_P(\mu) = \omega_n \left[(\alpha(\mu))^n \int_\mu^\infty \frac{d}{d\nu} \left(\frac{1}{\alpha(\nu)^n} \right) \frac{1}{n} \hat{\Phi}(\nu) \, \mathrm{d}\nu - P\mu \right], \tag{3.7}$$

where

$$\hat{\Phi}(\nu) = \Phi(\rho(\nu), \nu, ...) \tag{3.8}$$

and $\rho(\nu) := r_0'(1/\alpha(\nu))$.

Remark 3.3. *For $\mu = \lambda$ to minimise $\widehat{E}_P(\mu)$ we must have that*

$$\widehat{E}_P'(\lambda) = 0, \quad \widehat{E}_P''(\lambda) \geq 0. \tag{3.9}$$

Note that if the first condition in (3.9) *holds, then the corresponding radial map (see Proposition 3.1) is a cavitating equilibrium solution to our dead-load traction problem since it satisfies* (1.17), (1.18), (1.19).

Remark 3.4. *We note for later use that, in the variables $\mu, \rho(\mu)$ used above, the radial equilibrium equation can be written as (see also [19])*

$$\frac{d}{d\mu} \left[\Phi_{,1} \left(\rho(\mu), \mu, \mu, ... \right) \right] = (n-1) \frac{\left[\Phi_{,2} \left(\rho(\mu), \mu, \mu, ... \right) - \Phi_{,1} \left(\rho(\mu), \mu, \mu, ... \right) \right]}{\rho(\mu) - \mu}. \tag{3.10}$$

Remark 3.5. *It is of interest to note the following alternative expression for $\widehat{E}_P'$: suppose that $\mathbf{u}(\mathbf{x}, \mu)$ is a one-parameter family of solutions of the equilibrium equations* (1.2)*, parametrised by μ. We next calculate the derivative of the energy func-*

tional (1.8) *evaluated on such a family, at fixed* P*, with respect to* μ.

$$\begin{aligned}\frac{d}{d\mu}E_P(\mathbf{u}(\cdot,\mu)) &= \frac{d}{d\mu}\left[\int_B W(\nabla\mathbf{u}(\mathbf{x},\mu))\,d\mathbf{x} - \int_{\partial B} P\,\mathbf{u}(\mathbf{x},\mu).\mathbf{n}\,\mathbf{dS}\right] \\ &= \int_B \frac{\partial W}{\partial F^i_\alpha}(\nabla_{\mathbf{x}}\mathbf{u})\frac{\partial^2 u^i}{\partial x^\alpha\partial\mu}(\mathbf{x},\mu)\ d\mathbf{x} - \int_{\partial B} P\,\frac{\partial u^i}{\partial\mu}n^i\ dS \\ &= \int_B \frac{\partial}{\partial x^\alpha}\left[\frac{\partial u^i}{\partial\mu}\frac{\partial W}{\partial F^i_\alpha}(\nabla\mathbf{u})\right] - \int_{\partial B} P\,\frac{\partial u^i}{\partial\mu}n^i\ dS \\ &= \int_{\partial B}\frac{\partial u^i}{\partial\mu}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla\mathbf{u})n^\alpha - Pn^i\right]\ dS \qquad (3.11)\end{aligned}$$

In the case that $\mathbf{u}(\cdot,\mu)$ *(given by* (1.14)*) is a family of solutions of the radial equilibrium equation* (1.17) *and the natural boundary condition* (1.19)*, the above derivation still holds (see section 2), and if* $\mathbf{u}(\mathbf{x}) = \mu\mathbf{x}$ for $\mathbf{x}\in\partial B$*, the above expression yields*

$$\begin{aligned}\frac{d}{d\mu}E_P(\mathbf{u}(\cdot,\mu)) &= \int_{\partial B} x^i\left[\Phi_1(\rho(\mu),\mu,\mu,...)n^i - Pn^i\right]\ dS \\ &= \omega_n\left[\Phi_1(\rho(\mu),\mu,\mu,...) - P\right]. \qquad (3.12)\end{aligned}$$

Taking a second derivative and using Remark 3.4 then yields

$$\begin{aligned}\frac{d^2}{d\mu^2}E_P(\mathbf{u}(\cdot,\mu)) &= \frac{d}{d\mu}\ \omega_n\left[\Phi_{,1}(\rho(\mu),\mu,\mu,...) - P\right] \\ &= \omega_n(n-1)\frac{\left[\Phi_{,2}(\rho(\mu),\mu,\mu,...) - \Phi_{,1}(\rho(\mu),\mu,\mu,...)\right]}{\rho(\mu)-\mu}. \qquad (3.13)\end{aligned}$$

Remark 3.6. *It follows from the above calculations that:*

(i) If $\mu = \lambda$ *satisfies* $\widehat{E}'_P(\lambda) = 0, \widehat{E}''_P(\lambda) \neq 0$*, then the radial cavitation solution* (3.5) *satisfies* (1.18) *and the corresponding asymmetric homogeneous deformation (given by* (2.5), (2.6), (3.2)*) has equal energy by* (2.8)*. However, in this case,* (3.13) *shows that the corresponding asymmetric homogeneous deformation cannot be a solution of the Rivlin problem and hence cannot be a critical point of* (1.11).

(ii) If $\mu = \lambda$ *satisfies* $\widehat{E}'_P(\lambda) = \widehat{E}''_P(\lambda) = 0$ *it follows from* (2.8) *that the asymmetric homogeneous deformation* (2.5) *(with* $\alpha = \rho(\lambda), \beta = \lambda$*) is a solution of the Rivlin problem for this dead-load* P *(and corresponds to a critical point of* (1.11)*).*

3.2 Energy of corresponding asymmetric homogeneous deformations

As previously noted, each cavitating deformation considered in the last section gives rise to a corresponding homogeneous deformation (via (2.5), (2.6)) with equal energy (see (2.8)). We parametrise these axisymmetric homogeneous maps by μ and denote the corresponding energies by

$$\widehat{E}_a(\mu) := E_P(\mathbf{u}^h) = \frac{\omega_n}{n}\left[\hat{\Phi}(\mu) - P(\rho(\mu) + (n-1)\mu)\right],\ \mu \in (\lambda_{\text{crit}}, \infty), \tag{3.14}$$

where $\hat{\Phi}$ is given by (3.8). From this it follows that

$$\widehat{E}_a'(\mu) = \frac{\omega_n}{n}\left[\rho'(\mu)(\hat{\Phi}_1(\mu) - P) + (n-1)(\hat{\Phi}_2(\mu) - P)\right]. \tag{3.15}$$

Since $\rho(\mu) = \mu - \frac{\alpha(\mu)}{\alpha'(\mu)}$, it follows that $\rho'(\mu) = \frac{\alpha(\mu)\alpha''(\mu)}{(\alpha'(\mu))^2}$ and so we can also write (3.15) as

$$\widehat{E}_a'(\mu) = \frac{\omega_n}{n}\left(\frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}\nu}(\mu) - \frac{P}{(\alpha'(\mu))^2}((n-1)\alpha'(\mu))^2 + \alpha(\mu)\alpha''(\mu))\right). \tag{3.16}$$

Remark 3.7. *Note that minimisers of $\widehat{E}_a(\mu)$ (or, more generally, solutions of $\widehat{E}_a'(\mu) = 0$) need not necessarily correspond to solutions of the Rivlin problem* (1.12).

The next result which relates the derivative of the energy functional $\widehat{E}_p(\mu)$, (3.6), to that of $\widehat{E}_a(\mu)$ will be central to our main result.

Proposition 3.8. *For each $\mu \in (\lambda_{\text{crit}}, \infty)$,*

$$\hat{E}_P'(\mu) = \frac{d}{d\mu}\left(\alpha^n(\mu)\right)\int_\mu^\infty \frac{1}{\alpha^n(\nu)}\hat{E}_a'(\nu)\ d\nu. \tag{3.17}$$

Proof. We first calculate the derivative of $\widehat{E}_P(\mu)$ given by (3.7), noting that $\frac{\alpha(\mu)}{\alpha'(\mu)} = \alpha(\mu)r_0(1/\alpha(\mu)) - r_0'(1/\alpha(\mu))$, to yield

$$\widehat{E}_P'(\mu) = \omega_n\frac{\alpha'}{\alpha}\left\{n(\alpha(\mu))^n\int_\mu^\infty \frac{\alpha'(\nu)}{\alpha(\nu)^{n+1}}\hat{\Phi}(\nu)\,\mathrm{d}\nu - P[\mu - \rho(\mu)] - \hat{\Phi}(\mu)\right\}. \tag{3.18}$$

Performing an integration by parts on the first term in the integral yields

$$n\int_\mu^\infty \frac{\alpha'(\nu)}{\alpha(\nu)^{n+1}}\hat{\Phi}(\nu)\,\mathrm{d}\nu = \frac{1}{\alpha(\mu)^n}\hat{\Phi}(\mu) + \int_\mu^\infty \frac{1}{\alpha(\nu)^n}\frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}\nu}(\nu)\,\mathrm{d}\nu,$$

and substituting this into (3.18) then yields

$$\widehat{E}'_P(\mu) = \omega_n \left[\alpha'(\mu)(\alpha(\mu))^{n-1} \int_\mu^\infty \frac{1}{\alpha(\nu)^n} \frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}\nu}(\nu)\,\mathrm{d}\nu - P \right]. \tag{3.19}$$

By the Fundamental Theorem of Calculus, this is equivalent to

$$\begin{aligned} \widehat{E}'_P(\mu) = \omega_n \Bigg[\alpha'(\mu)(\alpha(\mu))^{n-1} \int_\mu^\infty \Bigg(& \frac{1}{\alpha(\nu)^n} \frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}\nu}(\nu) \\ & - \frac{P}{(\alpha'(\nu))^2(\alpha(\nu))^n}((n-1)(\alpha'(\nu))^2 + \alpha(\nu)\alpha''(\nu)) \Bigg) \,\mathrm{d}\nu \Bigg], \end{aligned} \tag{3.20}$$

which can be rewritten as

$$\begin{aligned} \widehat{E}'_P(\mu) = \omega_n \Bigg[\alpha'(\mu)(\alpha(\mu))^{n-1} \int_\mu^\infty \Bigg(\frac{1}{\alpha(\nu)^n} \Bigg[& \frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}\nu}(\nu) \\ & - \frac{P}{(\alpha'(\nu))^2}((n-1)(\alpha'(\nu))^2 + \alpha(\nu)\alpha''(\nu)) \Bigg] \Bigg) \,\mathrm{d}\nu \Bigg]. \end{aligned} \tag{3.21}$$

The result now follows from (3.16) and (3.21). □

Corollary 3.9. *If r_λ (given by (3.5))is a cavitating solution for the dead-load traction boundary value problem, then $\hat{E}'_P(\lambda) = 0$ (see Remark 3.3) and so*

$$\hat{E}'_P(\mu) = \frac{d}{d\mu}\left(\alpha^n(\mu)\right) \int_\mu^\lambda \frac{1}{\alpha^n(\mu)} \hat{E}'_a(\nu)\; d\nu \text{ for } \mu \in (\lambda_{\mathrm{crit}}, \infty). \tag{3.22}$$

Remark 3.10. *If we rewrite* (3.18) *using* (3.7) *and* (3.14) *we obtain*

$$\widehat{E}'_P(\mu) = \frac{\alpha'(\mu)n}{\alpha(\mu)}[\widehat{E}_P(\mu) - \widehat{E}_a(\mu)]. \tag{3.23}$$

This implies (see Remark 3.3) that if we have a cavitating solution of the radial equilibrium equations for the dead-load problem, $r_\lambda(R)$, then the total energy of the deformation is that of an asymmetric homogeneous deformation whose stretches are $\rho(\lambda), \lambda, ..., \lambda$, as previously shown in (2.8).

4 Energy minimising properties

In this section we will show that for each $P > 0$ that if there is a cavitating deformation $\mathbf{u}_{cav}$ which minimises (3.6), then necessarily there exists an asymmetric homogeneous deformation $\mathbf{u}^h$ which has less energy. Recall that we have so far shown that for any deformation $\mathbf{u}_{cav}$ which minimises (3.6) the total energy is equal to the energy of a corresponding asymmetric homogeneous deformation. Since (1.11) has a minimiser for all P, if we show that this asymmetric homogeneous deformation is not a minimiser of (1.11), it will then follow that there must necessarily exist an asymmetric homogeneous deformation $\mathbf{u}^h$ which has less energy than that of $\mathbf{u}_{cav}$.

Remark 4.1. *For fixed P, from differentiating equation* (3.23)*, it follows that if $\mu = \lambda$ minimises $\widehat{E}_P(\mu)$, then*

$$\widehat{E}''_P(\lambda) = -\frac{\alpha'(\lambda)n}{\alpha(\lambda)}\widehat{E}'_a(\lambda).$$

It therefore follows immediately that if $\widehat{E}''_p(\lambda) \neq 0$, then $\widehat{E}'_a(\lambda) \neq 0$ and so $\mu = \lambda$ cannot be a local minimiser of $\widehat{E}_a(\lambda)$. Hence, if we suppose further that $\widehat{E}'_P(\lambda) = 0$ then, using Remark 3.10, there must exist $\hat{\mu}$ such that

$$\widehat{E}_a(\hat{\mu}) < \widehat{E}_P(\lambda) = \widehat{E}_a(\lambda), \tag{4.1}$$

i.e., there exists an asymmetric homogeneous deformation with less energy than the symmetric cavitation solution. In particular, any global minimiser of the functional (1.11) *for the Rivlin problem has less energy than this radial cavitation solution.*

This observation, will yield the main result of this paper in Theorem 4.2. It remains to eliminate the possibility that $\widehat{E}'_p(\lambda) = 0$ and $\widehat{E}''_p(\lambda) = 0$. To this end, we will henceforth assume that the following non-degeneracy condition holds:

(**N**) (**Non-degeneracy condition**): we assume that $\widehat{E}''_a(\mu)$ vanishes at most at a finite number of points on the interval $[1, \infty)$.

Theorem 4.2. *If a radially symmetric cavitating deformation* $\mathbf{u}$ *given by* (1.14), (3.5) *and corresponding to $\mu = \lambda$, minimises the radial energy* (3.6)*, then there exists an asymmetric homogeneous deformation $\mathbf{u}^h$ such that $E(\mathbf{u}) > E(\mathbf{u}^h)$.*

Proof. First observe that by Remark 4.1, if $\widehat{E}''_P(\lambda) > 0$ then $\widehat{E}'_a(\lambda) \neq 0$ and the result follows. Hence it remains to treat the case in which $\widehat{E}''_P(\lambda) = 0$. It is in this case that the non-degeneracy condition (N) above will play a role.

For $\widehat{E}_P$ and $\widehat{E}_a$ given by (3.7) and (3.14), we show that if $\mu = \lambda$ is a minimiser of $\widehat{E}_P(\mu)$ then there exists a $\tau \in (0, \infty)$ such that $\widehat{E}_P(\lambda) > \widehat{E}_a(\tau)$ and τ minimises $\widehat{E}_a$.

To prove this result it suffices to show that $\mu = \lambda$ is not a minimiser of $\widehat{E}_a$. Suppose for a contradiction that $\mu = \lambda$ is simultaneously a minimiser of (3.7) and (3.14).

From Corollary 3.9 it follows that

$$\widehat{E}'_P(\mu) = -\omega_n(n-1)\alpha'(\mu)(\alpha(\mu))^{n-1} \int_\lambda^\mu \frac{1}{\alpha(\nu)^n} \widehat{E}'_a(\nu)\,\mathrm{d}\nu. \tag{4.2}$$

Using (N) we know that there is an $\epsilon > 0$ such that $\widehat{E}'_a(\nu) > 0$ for all $\nu \in (\lambda, \lambda + \epsilon)$. As $\widehat{E}'_a(\nu) > 0$ for all $\nu \in (\lambda, \lambda + \epsilon)$ we see that $\widehat{E}'_P(\mu) < 0$ for all $\mu \in (\lambda, \lambda + \epsilon)$, contradicting the fact that $\mu = \lambda$ minimises $\widehat{E}_P$. $\square$

5 Concluding Remarks

In the case of incompressible materials, any admissible deformation $\mathbf{u}$ must satisfy the condition $\det(\nabla \mathbf{u}) = 1$. Next note that the kinematic constraint of incompressibility implies that the principle stretches must satisfy $v_1 ... v_n = 1$, which implies that every radially symmetric deformation must have the form

$$\mathbf{u}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x}, \quad \text{where } r(R) = (R^n + A^n)^{\frac{1}{n}}, \tag{5.1}$$

where $R := |\mathbf{x}|$ and A is the radius of the cavity formed.

In [22], Abeyaratne and Hou characterise their deformations using the radial boundary stretch $r'(1)$ rather than the tangential boundary stretch $\frac{r(1)}{1} = r(1)$. In the incompressible case, if we perform a corresponding change of independent variable in (4.2) from $\nu = \frac{r}{R}$ to $\frac{1}{\nu^{n-1}} = r'$, then we obtain equation (32) in [22].

We now show that the main results of [22] for incompressible materials follow from our arguments and our main result Theorem 4.2.

The version of Green's divergence identity (2.1) for incompressible materials is

$$\mathrm{Div}\left[W(\nabla\mathbf{u})\mathbf{x} + \left(\frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}) - p\,(\mathrm{adj}\nabla\mathbf{u})\right)^T (\mathbf{u} - \nabla\mathbf{u}\,\mathbf{x})\right] = nW(\nabla\mathbf{u}), \tag{5.2}$$

where p is an arbitrary hydrostatic pressure. The conclusions of section 2 now follow in the incompressible case on replacing (2.1) by (5.2). Next replace $W(\mathbf{F})$ by $W(\mathbf{F}) + p(\det\nabla\mathbf{u} - 1)$ and $\Phi(v_1, ..., v_n)$ by $\Phi(v_1, ..., v_n) + p(v_1 ... v_n - 1)$ in the arguments of section 3. Note that in this incompressible case, the function corresponding to the r_0 given in Proposition 3.1 is then given by $r_0(R) = (R^n + 1)^{\frac{1}{n}}$ and $\alpha(\mu) = (\mu^n - 1)^{\frac{1}{n}}$.

We next recall that our results for compressible materials were obtained under the assumption that the stored energy function W satisfies the non-degeneracy condition (N) and to obtain the results of [22] we need to replace this condition by its incompressible counterpart. We recall that this condition is used to ensure that if $\mu = \hat{\mu}$ is a local minimum of $\hat{E}_a(\mu)$ and $\hat{E}'_a(\hat{\mu}) = \hat{E}''_a(\hat{\mu}) = 0$, then $E'_a(\mu) > 0$ on some interval $(\hat{\mu}, \hat{\mu} + \epsilon), \epsilon > 0$ and similarly that $E'_a(\mu) < 0$ on some interval $(\hat{\mu} - \epsilon, \mu)$. A related non-degeneracy condition arises in section 5 of [22] (see their condition (A)).

In the incompressible case, it can be shown that Mooney-Rivlin materials and certain Ogden materials satisfy the incompressible version of (N). However, it would be of interest to identify general conditions which guarantee that a given stored energy function satisfies these non-degeneracy conditions.

Hence, the results of the current paper apply to both incompressible and compressible isotropic hyperelastic materials and demonstrate that if a unit ball of an elastic material is subjected to a uniform dead-load on the boundary, it is energetically favourable for it to bifurcate into a asymmetric homogeneous deformation rather than into a radially symmetric cavitated deformation. These results demonstrate that the radial cavitation solution is never a *global* minimiser of the energy but do not rule out the possibility that it is a still a local minimiser in a general class of possibly nonsymmetric perturbation. The symmetrisation arguments in [26] may be helpful in studying this problem.

In the compressible case it can be shown that (N) is satisfied by separable stored energy functions and stored energy function in [27] given by

$$\Phi(v_1, v_2, v_3) = (v_1^2 + v_2^2 + v_3^2) + h(v_1 v_2 v_3),$$

where

$$h(d) = (ad^2 - 2(a+1)d + b)$$

for $d \geq 1$ and $a, b \geq 0$ are constants[5]. It is currently unclear to us whether compressible Mooney-Rivlin and Ogden materials satisfy (N).

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