Order-Invariance of Cylindrical Algebraic Decomposition via Triangular Decomposition

David J. Wilson

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Introduction

These notes are a product of research seminars held at the University of Bath in Autumn 2012. The research group of Bath consisted of James Davenport, Russell Bradford, Matthew England, Acyr Locatelli and David Wilson. They were joined by Scott McCallum of Macquarie University who was at Bath as a visiting researcher.

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1 Importance of Order-Invariance

Practical uses of cylindrical algebraic decomposition usually depend on sign-invariance as defined below:

Definition 1.
Let $F \subseteq \mathbf{R}[x_1, \ldots, x_n]$. A set $X \subseteq \mathbf{R}^n$ is said to be **F-sign-invariant** if for every $f \in F$ the sign of F is constant (strictly positive, strictly negative, or zero) on X. A cylindrical algebraic decomposition $\mathcal{D}$ of $\mathbf{R}^n$ is **F-sign-invariant** if each cell $D \in \mathcal{D}$ is F-sign-invariant.

We similarly say a set $X \subseteq \mathbf{C}^n$ is **F-sign-invariant** if for every $f \in F$ the value of f, when $x_1, \ldots, x_n$ are thought of as complex variables, is uniformly zero or non-zero. A cylindrical decomposition $\mathcal{C}$ of $\mathbf{C}^n$ is **F-sign-invariant** if each cell $C \in \mathcal{C}$ is F-sign-invariant.

However, as documented in [McC85], in general a stronger condition is needed for proofs by induction over a cylindrical algebraic decomposition. This is the concept of order-invariance which is defined as follows:

Definition 2.
Let $f \in \mathbf{R}[x_1, \ldots, x_n]$ and let $\mathbf{a} \in \mathbf{R}^n$. We say the **order** of f at $\mathbf{a}$ is the least non-negative integer k such that some partial derivative of f of order k does not vanish at $\mathbf{a}$.

Let $F \subseteq \mathbf{R}[x_1, \ldots, x_n]$. A set $X \subseteq \mathbf{R}^n$ is said to be **F-order-invariant** if for every $f \in F$ the order of F is constant on X. A cylindrical algebraic decomposition $\mathcal{D}$ of $\mathbf{R}^n$ is **F-order-invariant** if each cell $D \in \mathcal{D}$ is F-order-invariant.

It is clear that this is stronger than sign-invariance, and the concept allows for inductive proofs on the validity of simplified projection operators (for example [McC85],[Bro01]) from Collins' original ([Col75]) method. In short, lifting over an order-invariant cylindrical algebraic decomposition produces a sign-invariant cylindrical algebraic decomposition (whereas lifting over a sign-invariant cylindrical algebraic decomposition does not guarantee sign-invariance). Order-invariance and sign-invariance are very different, as shown in the following example.

Example 1.
Consider the equation $f(x, y) = y(x^2 + y^2)$. This has the graph shown in Figure 1. We see that f has order 3 at the origin: the first derivatives $f_x = 2yx$ and $f_y = x^2 + 3y^2$ both vanish, as do the second derivatives $f_{xx} = 2y$, $f_{xy} = 2x$, and $f_{yy} = 6y$. However, away from the origin on the x-axis, say at $(1, 0)$, the order is 1 (f_y does not vanish). The minimal sign-invariant cylindrical algebraic decomposition (3 cells, split by the x-axis) is therefore not the same as the minimal order-invariant cylindrical algebraic decomposition (9 cells, due to adding in $x = 0$ to the sign-invariant cad).

2 Order Invariance of Maple CAD

Order invariance in classical cylindrical algebraic decomposition is a by-product of sign-invariance of discriminants, resultants and coefficients. Initial experiments with Maple suggested that the Maple CAD (based on [CMXY09]) may

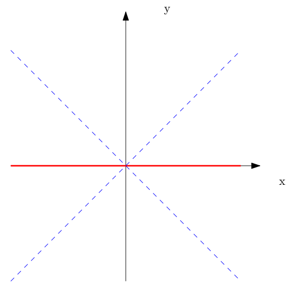

Figure 1: Graphical representation of $f(x, y) = y(x^2 + y^2)$ with the 'shadows' of $x = \pm iy$

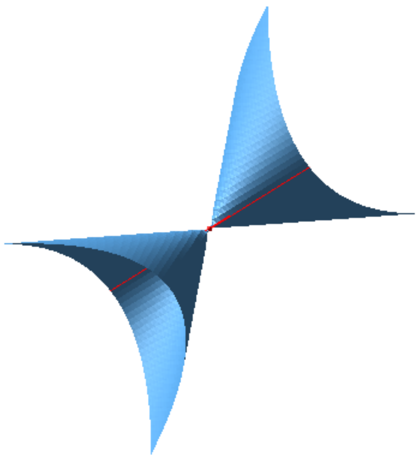

Figure 2: Graphical representation of $f(x, y, z) = z^2 - xy$ with the x-axis highlighted

always produce an order-invariant CAD. For instance, inputting Example 1 into Maple (or, indeed, QEPCAD) produces an order-invariant CAD.

However, the following example disproved that conjecture:

Example 2.
Let $f(x, y, z) = z^2 - xy$, the graph of which is shown in Figure 2. Note this is related to the Whitney Umbrella, $\widehat{f}(x, y, z) = z^2 - y^2x$ but is slightly simpler.

Then a subset of the solutions to f is the x-axis: $\{(a, 0, 0) \mid x \in \mathbf{R}\}$. If $a \neq 0$ then $f_y = -a \neq 0$ and so f has order 1. However at the origin $f_x, f_y, f_z = 0$ and $f_{zz} \neq 0$ so f has order 2. Therefore an order-invariant cylindrical algebraic decomposition must distinguish the origin as an individual point on the axis.

We can input f into Maple and compute both a cylindrical decomposition of $\mathbf{C}^3$ and a cylindrical algebraic decomposition of $\mathbf{R}^3$. The former produces the output shown in Figure 3, which we can see does not distinguish the origin as a special point.

Similarly we can input f into the `CylindricalAlgebraicDecompose` algorithm and obtain the output shown in Figure 4. Once again we note that the

$$\begin{cases} \begin{cases} 1 & y = 0 \\ \begin{cases} 1 & xy - z^2 = 0 \\ 1 & \text{otherwise} \end{cases} & \text{otherwise} \end{cases} & z = 0 \\ \begin{cases} 1 & y = 0 \\ \begin{cases} 1 & xy - z^2 = 0 \\ 1 & \text{otherwise} \end{cases} & \text{otherwise} \end{cases} & \text{otherwise} \end{cases}$$

Figure 3: Output of `CylindricalDecompose` in MAPLE for $f = z^2 - xy$

origin is not a distinguished point, therefore the Maple cylindrical algebraic decomposition algorithm does not produce an order-invariant cylindrical algebraic decomposition.

3 Well-Oriented Sets of Polynomials

Looking at the examples we have found where MAPLE fails to produce an order-invariant CAD reveals that they are polynomials which are not **well-oriented**, as defined in [CJ98].

Definition 3 ([CJ98])**.**
Let P be the McCallum projection operator (as defined in [McC85] and [McC88]) and let $F \subseteq \mathbf{Z}[x_1, \ldots, x_n]$. Then F is said to be **well-oriented** if whenever $n > 1$ the following conditions hold:

WO 1. For every $f \in \text{prim}(F)$ (the set of primitive parts of polynomials in F with positive degree in x_n), $f(\mathbf{a}, x_n) = 0$ for at most a finite number of points $\mathbf{a} \in \mathbf{R}^{n-1}$; and

WO 2. $P(F)$ is well-oriented.

McCallum even suggests a stronger condition that can be considered, that of **very well-orientedness** where the allowance of a finite set of nullifying points is removed.

Well-orientedness is needed for both the McCallum ([McC85]) and Brown-McCallum ([Bro01]) projection operators to produce an order-invariant cylindrical algebraic decomposition. This fact, along with experimentation, leads to the following conjecture:

Conjecture 1.
Let $F \subseteq \mathbf{R}[x_1, \ldots, x_n]$. Then the MAPLE *cylindrical algebraic decomposition algorithm produces an order-invariant cylindrical algebraic decomposition if F is well-oriented with respect to the given projection order.*

Remark 1.
This reverse implication of the conjecture does not hold. There exists non-well-oriented sets of polynomials for which Maple still produces an order-invariant cylindrical algebraic decomposition.

The calculation of discriminants and resultants (in a similar fashion to the McCallum projection operator) hints that this conjecture has potential validity. However, the algorithm needs to be finely studied to see if a proof is possible.

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\begin{cases}
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\begin{cases}
[regular\_chain, [[-1,-1],[-1,-1],[-2,-2]]] & x < \frac{z^2}{y} \\
[regular\_chain, [[-1,-1],[-1,-1],[-1,-1]]] & x = \frac{z^2}{y} \\
[regular\_chain, [[-1,-1],[-1,-1],[0,0]]] & \frac{z^2}{y} < x
\end{cases} & y < 0 \\
[regular\_chain, [[-1,-1],[0,0],[0,0]]] & y = 0 \\
\begin{cases}
[regular\_chain, [[-1,-1],[1,1],[0,0]]] & x < \frac{z^2}{y} \\
[regular\_chain, [[-1,-1],[1,1],[1,1]]] & x = \frac{z^2}{y} \\
[regular\_chain, [[-1,-1],[1,1],[2,2]]] & \frac{z^2}{y} < x
\end{cases} & 0 < y
\end{cases} & z < 0 \\
\begin{cases}
\begin{cases}
[regular\_chain, [[0,0],[-1,-1],[-1,-1]]] & x < \frac{z^2}{y} \\
[regular\_chain, [[0,0],[-1,-1],[0,0]]] & x = \frac{z^2}{y} \\
[regular\_chain, [[0,0],[-1,-1],[1,1]]] & \frac{z^2}{y} < x
\end{cases} & y < 0 \\
[regular\_chain, [[0,0],[0,0],[0,0]]] & y = 0 \\
\begin{cases}
[regular\_chain, [[0,0],[1,1],[-1,-1]]] & x < \frac{z^2}{y} \\
[regular\_chain, [[0,0],[1,1],[0,0]]] & x = \frac{z^2}{y} \\
[regular\_chain, [[0,0],[1,1],[1,1]]] & \frac{z^2}{y} < x
\end{cases} & 0 < y
\end{cases} & z = 0 \\
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\begin{cases}
[regular\_chain, [[1,1],[-1,-1],[-2,-2]]] & x < \frac{z^2}{y} \\
[regular\_chain, [[1,1],[-1,-1],[-1,-1]]] & x = \frac{z^2}{y} \\
[regular\_chain, [[1,1],[-1,-1],[0,0]]] & \frac{z^2}{y} < x
\end{cases} & y < 0 \\
[regular\_chain, [[1,1],[0,0],[0,0]]] & y = 0 \\
\begin{cases}
[regular\_chain, [[1,1],[1,1],[0,0]]] & x < \frac{z^2}{y} \\
[regular\_chain, [[1,1],[1,1],[1,1]]] & x = \frac{z^2}{y} \\
[regular\_chain, [[1,1],[1,1],[2,2]]] & \frac{z^2}{y} < x
\end{cases} & 0 < y
\end{cases} & 0 < z
\end{cases}
$$

Figure 4: Output of `CylindricalAlgebraicDecompose` in Maple for $f = z^2 - xy$

References

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