



Citation for published version:

Di Francesco, M & Donatelli, D 2010, 'Singular convergence of nonlinear hyperbolic chemotaxis systems to keller-segel type models', *Discrete and Continuous Dynamical Systems - Series B*, vol. 13, no. 1, pp. 79-100. <https://doi.org/10.3934/dcdsb.2010.13.79>

DOI:

[10.3934/dcdsb.2010.13.79](https://doi.org/10.3934/dcdsb.2010.13.79)

Publication date:

2010

Document Version

Publisher's PDF, also known as Version of record

[Link to publication](#)

University of Bath

Alternative formats

If you require this document in an alternative format, please contact:
openaccess@bath.ac.uk

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

SINGULAR CONVERGENCE OF NONLINEAR HYPERBOLIC CHEMOTAXIS SYSTEMS TO KELLER–SEGEL TYPE MODELS

MARCO DI FRANCESCO

Sezione di Matematica per l'Ingegneria, Dipartimento di Matematica Pura ed Applicata
Università degli Studi dell'Aquila, Facoltà di Ingegneria
Piazzale E. Pontieri, I-67040 - Monteluco di Roio - L'Aquila, Italy

DONATELLA DONATELLI

Dipartimento di Matematica Pura ed Applicata
Università degli Studi dell'Aquila
Via Vetoio, Loc. Coppito - I-67010 - L'Aquila, Italy

(Communicated by Christian Schmeiser)

ABSTRACT. In this paper we deal with diffusive relaxation limits of nonlinear systems of Euler type modeling chemotactic movement of cells toward Keller–Segel type systems. The approximating systems are either hyperbolic–parabolic or hyperbolic–elliptic. They all feature a nonlinear pressure term arising from a *volume filling effect* which takes into account the fact that cells do not interpenetrate. The main convergence result relies on energy methods and compensated compactness tools and is obtained for large initial data under suitable assumptions on the approximating solutions. In order to justify such assumptions, we also prove an existence result for initial data which are small perturbation of a constant state. Such result is proven via classical Friedrichs's symmetrization and linearization. In order to simplify the coverage, we restrict to the two-dimensional case with periodical boundary conditions.

1. Introduction. This paper deals with diffusive relaxation limits for the nonlinear hyperbolic model describing chemotactic movement of cells, also known as the *persistence and chemotaxis* model,

$$\begin{cases} \partial_\tau \rho + \nabla \cdot (\rho v) = 0 \\ \partial_\tau v + v \cdot \nabla v + \nabla g(\rho) = \nabla c - dv \\ \sigma \partial_\tau c = \Delta c + \alpha \rho - \beta c, \end{cases} \quad (1)$$

with α, β, d, σ positive constants. The function $g(\rho)$ is taken of the form $g(\rho) = \rho^\gamma$ with $\gamma > 0$, we shall discuss this choice later on in this introduction. The model (1) has been introduced and motivated very precisely in [1], whereas similar models have been also discussed in [22, 29, 12, 20]. We shall briefly summarize the biological motivations behind (1) by framing them in the general context of PDE systems describing chemotactical phenomena.

2000 *Mathematics Subject Classification.* Primary: 35L99, 35B25, 92C17; Secondary: 35Q80, 76N10.

Key words and phrases. Singular convergence, diffusive relaxation, chemotaxis, nonlinear hyperbolic systems, Keller Segel model, nonlinear diffusion, volume filling effect.

The analysis of partial differential equations modeling chemotaxis goes back to the work of Keller and Segel [32], who proposed a macroscopic model for aggregation of cellular slime molds, and to the earlier related work of Patlak [44], who derived similar models with applications to the study of long-chain polymers. In the successive decades, the term *chemotaxis* has been used to represent the dynamics of several biological systems (such as the bacteria *Escherichia Coli*, or the amoebae *Dictyostelium Discoideum*, or endothelial cells of the human body responding to angiogenic factors secreted by a tumor) in which the motion of a species is biased by the gradient of a certain chemical substance. Typically these models consist of a system of drift–diffusion equations of the form

$$\begin{cases} \rho_t = \Delta\rho - \nabla \cdot (\rho\chi(\rho, c)\nabla c) \\ c_t = \Delta c + r(\rho, c), \end{cases} \quad (2)$$

with diffusion terms modeling random motion for the density ρ of the individuals (cells, bacteria and so on) and for the concentration c of the *chemoattractant* (the chemical substance responsible of the chemotactical movement), first order drift terms modeling chemotactical aggregation and zero order reaction terms in the equation for the chemoattractant. The coefficient $\chi(\rho, c)$ is called *chemotactical sensitivity*. The simplest model combining diffusion and chemotaxis is the well known parabolic–elliptic Patlak–Keller–Segel system (or simply Keller–Segel system)

$$\begin{cases} \rho_t = \Delta\rho - \nabla \cdot (\rho\nabla c) \\ 0 = \Delta c + \rho, \end{cases} \quad (3)$$

which has the interesting mathematical feature of producing smooth solutions for small initial norms (in the appropriate space) and blow-up (in the form of concentration to deltas) for large initial norms. The rigorous analysis of such mathematical issue (also extended to fully parabolic systems and to more complex models) has attracted the attention of many mathematicians in the last decades. We mention the pioneering works of Jäger–Luckaus [28], Nagai [42], Herrero–Velazquez [27] among others. The existence vs. blow-up problem in two space dimensions for the classical Keller–Segel model (3) has been completely solved in [19], where the authors proved that if the initial mass is less than a threshold value m^* (depending on the coefficient χ) then the solution exists globally in time in L^1 , whereas if the initial mass is larger than m^* then the solution blows up in a finite time. A more complete presentation of this issue is provided in [3]. We refer to the surveys [24, 25] for a complete and detailed description of the literature of this topic.

In the last years, some authors [31, 5, 35] have proposed variants of Keller–Segel type models featuring global existence of solutions no matter how large the initial mass is, obtained by replacing the linear diffusion term in the equation for the population density by a *degenerate nonlinear diffusion* term with super-linear growth for large densities. This choice can be motivated by taking into account the fact that cells do not interpenetrate (that is, they are full bodies with nonzero volume) and therefore diffusion is supposed to inhibit singular aggregation effects when the density is very high. We mention here that other authors proposed the use of a nonlinear chemotaxis coefficient $\rho\chi(\rho)$ which attains the value zero when the population density ρ reaches a fixed maximal value – see for instance [45, 4, 2, 26] – being this choice motivated by the fact that individuals stop aggregating when the density is too high. In both cases, in the resultant model overcrowding of cells

(concentration to deltas for the cells density ρ) is prevented independently on any initial parameter.

In the last years, several authors have started to describe biological systems with chemotaxis from a hydrodynamical point of view, i. e. via nonlinear hyperbolic systems of Euler type, see in particular [1, 22, 29, 12, 20]. The models obtained are of the form of system (1), where the chemotactical force ∇c in (1) and the pressure contribute to balance the rate of change of the momentum. Moreover, our model (1) features a friction term modeling the drag between cells and the substrate material (some authors also considered models with a linear viscous term). In this framework, the nonlinear pressure term $g(\rho)$ in (1) plays the role of the diffusive one in the drift–diffusion equation. Therefore, one can interpret the overcrowding–preventing effect described before (sometimes referred to as *volume filling effect*) by thinking of the cellular matter as a medium with *limited compressibility*, i. e. closely packed cells exhibit a limited amount of resistance to compression. In this sense, a reasonable choice of a pressure $g(\rho)$ is a function of the form $g(\rho) = \rho^\gamma$, $\gamma > 0$. Such an expression also has the advantage of modeling absence of stresses for low densities (see [1] for a more detailed description).

In this paper we want to contribute to the problem of establishing a rigorous mathematical link between the system (1) and several Keller Segel type models of the form (2) in terms of diffusive relaxation limits. A typical example of diffusive scaling on the system (1) that we shall consider (see subsection 2.1) is the following

$$d = \frac{1}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon}, \quad v^\varepsilon(x, t) = \frac{1}{\varepsilon} v\left(x, \frac{t}{\varepsilon}\right)$$

which transforms (1) into the following rescaled system

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon v^\varepsilon) = 0 \\ \varepsilon^2 [\partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon] + \nabla g(\rho^\varepsilon) = \nabla c^\varepsilon - v^\varepsilon \\ \varepsilon \partial_t c^\varepsilon = \Delta c^\varepsilon + \alpha \rho^\varepsilon - \beta c^\varepsilon. \end{cases} \quad (4)$$

Formally, as $\varepsilon \rightarrow 0$, we expect the solution $(\rho^\varepsilon, v^\varepsilon, c^\varepsilon)$ to system (4) to behave like the solution (ρ^0, v^0, c^0) to the drift–diffusion system of Keller–Segel type

$$\begin{cases} \partial_t \rho^0 + \nabla \cdot (\rho^0 \nabla (c^0 - g(\rho^0))) = 0 \\ \Delta c^0 + \alpha \rho^0 - \beta c^0 = 0, \end{cases} \quad (5)$$

where the loss of the persistence term in the equation for the momentum yields a constitutive law for the velocity $v^0 = \nabla c^0 - \nabla g(\rho^0)$ (which can be considered as an equivalent of the Darcy law in [36]).

A way to understand the meaning of this phenomena is to consider it as the large time behaviour of dissipative nonlinear hyperbolic systems and to look at the asymptotic profile as the relaxed equilibrium. This is the case for many relevant situations in mathematical physics and applied mathematics. Singular limits with a structure similar to (4) have been analyzed by Marcati and Milani [36]. In that paper they investigate the porous media flow as the limit of the Euler equation in $1 - D$, later generalized by Marcati and Rubino [39] to the $2 - D$ case. Relaxation phenomena of the same nature appear also in the zero relaxation limits for the Euler–Poisson model for semiconductors devices and they were investigated by Marcati and Natalini [37, 38] in the $1 - D$ case and by Lattanzio and Marcati [34] in the multi-D case. We remark that the rigorous justification of diffusive relaxation limits appears also in the context of kinetic models for chemotaxis, see [7, 8]. For a general

overview of the theory of the singular limits see the survey [17] and the paper [18], where the theory is completely set up.

To perform the relaxation limit we follow the same techniques developed in [36, 39, 18] (among others), which are crucially based on the method of compensated compactness of Tartar and Murat (see [46, 47, 41]) combined with the Young measures associated to the relaxing sequence ρ^ε (see [46, 13, 14, 15, 16]). Throughout the whole paper, we shall restrict ourselves to the case of two space dimensions, which is also the most treated case in the literature concerning Keller–Segel type systems. Moreover, for the sake of simplicity we shall work on the 2-dimensional torus. We shall prove that this singular limit can be rigorously justified as far as the new time variable τ stays in a bounded interval $[0, T]$ for an arbitrary T and provided that certain a priori assumptions holds for the solution to (4) (see assumption 1 below). These a priori assumptions are usual in the framework of relaxation limits for nonlinear hyperbolic systems (see also [34], [39]) and they don't include any smallness assumption on the initial conditions. The main additional difficulty in our estimates (with respect to the afore mentioned references) lies on the control of the gradient of the chemoattractant c , which is responsible for the aggregation of cells and therefore produces an ‘anti-dissipative’ term in the total energy of the system (contrary to what happens, for instance, in hydrodynamical models for semiconductors). The rigorous statements of these results are contained in Theorem 4.2.

In order to produce a nontrivial class of solutions to the nonlinear hyperbolic system (1) which relax toward a Keller–Segel type model after a proper rescaling, we shall also provide an existence theorem for the approximating system (4) and prove the uniform estimates needed to justify the assumptions (1) in case of initial densities ρ_0 which are small perturbation of an arbitrary non zero constant state (see Theorem 5.1). This result is achieved by means of the classical Friedrichs’ symmetrization technique and by a linearization argument, see [21, 30]. We remark that, in many of the estimates performed here, the pressure term need not to be of the form $g(\rho) = \rho^\gamma$. Indeed, some of the estimates proven are still valid if one considers a logarithmic pressure $g(\rho) = \log \rho$, which corresponds to a linear diffusion term in the limit problem (2) (this fact is not in contradiction with the blow-up of the density in the limit problem with linear diffusion, see the Remark 6). However, while considering the alternative scaling introduced in subsection 2.2 (where the limit is the classical Keller–Segel system (3)), such an expression for the pressure seems to be essential in order to achieve the needed estimates no matter how large the initial mass is, in a very similar fashion to what happens in [5]. We remark that our convergence results hold on an arbitrary time interval. Therefore, at least in the case of the second scaling treated in section 2.2 (where the expression $g(\rho) = \rho^\gamma$ is crucial in order to achieve the desired estimates), our result can be seen as a new interpretation of the overcrowding–preventing effect due to the power–like expression of the pressure. More precisely, the global smoothness of the limit density ρ^0 (and the absence of concentration to deltas for all times of ρ^0 as a byproduct) can be obtained as a consequence of our relaxation result, alternatively to the more direct proof developed in [31, 5].

The paper is organized as follows. In section 2 we state the three different scalings we shall deal with. In section 3 we perform the uniform estimate we need in order to prove the main convergence theorem. In section 4 we prove the main convergence theorem for large data under the a priori assumption 1 by means of compensated

compactness and Minty's argument. In section 5 we prove an existence theorem for the approximating rescaled system in order to provide a class of solutions satisfying the basic assumptions 1.

2. Preliminaries and rescalings. We consider the following nonlinear mixed hyperbolic–parabolic system modelling *persistence and chemotaxis*

$$\begin{cases} \partial_\tau \rho + \nabla \cdot (\rho v) = 0 \\ \partial_\tau v + v \cdot \nabla v + \nabla g(\rho) = \nabla c - dv \\ \sigma \partial_\tau c = \Delta c + \alpha \rho - \beta c \end{cases} \quad (6)$$

where α, β, d, σ are nonnegative constants. The system (6) is endowed with the following 1–periodic initial data

$$\rho(0, x) = \rho_0(x), \quad v(0, x) = v_0(x), \quad c(0, x) = c_0(x)$$

The nonlinear function $g(\rho)$ has the form

$$g(\rho) = \rho^\gamma, \text{ for some } \gamma > 0.$$

Remark 1. The nonlinear pressure $g(\rho)$ grows faster than the critical pressure (in the sense of the classical 2 dimensional Keller–Segel system): there exists $\kappa > m/4\pi =: \kappa^*$ and $U > 0$ such that for all $\rho > U$, $g(\rho) \geq \kappa \log \rho$ (see [5]).

Some of the results contained in the present paper hold in any space dimension n , whereas some of them are true only in the case $n = 2$. In order to simplify the coverage, we shall always restrict ourselves to the latter case. In the sections 2, 3 and 4 we shall not deal with the existence theory of (6), whereas we shall work under the following basic assumption.

Assumption 1. *There exists a global solution (ρ, v, c) to (6), smooth enough in order to justify the estimates contained in section 3 and such that*

- (A1) *the total mass $M = \int \rho dx$ is conserved,*
- (A2) *$\rho(x, t) \geq k > 0$,*
- (A3) *$(\rho, \rho v) \in L^\infty(\mathbb{T}^2 \times [0, +\infty))$.*

Let us now explain in detail the relaxation limits we want to perform. We shall deal with three different asymptotic regimes for (6), corresponding to small parameter limits of three different types of scaling.

2.1. First scaling: Large time and large damping. For a fixed constant $\varepsilon > 0$ we consider the large damping rate $d = \frac{1}{\varepsilon}$ in (6). Then, we introduce the fast time variable

$$\tau = \frac{t}{\varepsilon},$$

and the new independent variables

$$v^\varepsilon(x, t) = \frac{1}{\varepsilon} v(x, \tau), \quad \rho^\varepsilon(x, t) = \rho(x, \tau), \quad c^\varepsilon(x, t) = c(x, \tau). \quad (7)$$

Moreover, we fix $\sigma = 1$ in the third equation. Then, system (6) in the new variables reads

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon v^\varepsilon) = 0 \\ \varepsilon^2 [\partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon] + \nabla g(\rho^\varepsilon) = \nabla c^\varepsilon - v^\varepsilon \\ \varepsilon \partial_t c^\varepsilon = \Delta c^\varepsilon + \alpha \rho^\varepsilon - \beta c^\varepsilon. \end{cases} \quad (8)$$

The formal limit as $\varepsilon \rightarrow 0$ is given by the parabolic–elliptic system

$$\begin{cases} \rho_t^0 + \nabla \cdot (\rho^0 \nabla (c^0 - g(\rho^0))) = 0 \\ \Delta c^0 + \alpha \rho^0 - \beta c^0 = 0. \end{cases} \quad (9)$$

2.2. Second scaling: Large time and large damping with Poisson coupling.

A simplified version of (6), namely with $\beta = 0$ and $\sigma = 0$, is given by the following system

$$\begin{cases} \partial_\tau \rho + \nabla \cdot (\rho v) = 0 \\ \partial_\tau v + v \cdot \nabla v + \nabla g(\rho) = \nabla c - dv \\ 0 = \Delta c + \alpha \rho. \end{cases} \quad (10)$$

By performing the same scaling as before, namely

$$\tau = \frac{t}{\varepsilon}, \quad v^\varepsilon(x, t) = \frac{1}{\varepsilon} v(x, \tau), \quad \rho^\varepsilon(x, t) = \rho(x, \tau), \quad c^\varepsilon(x, t) = c(x, \tau), \quad (11)$$

and by putting $d = \frac{1}{\varepsilon}$, we obtain

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon v^\varepsilon) = 0 \\ \varepsilon^2 [\partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon] + \nabla g(\rho^\varepsilon) = \nabla c^\varepsilon - v^\varepsilon \\ 0 = \Delta c^\varepsilon + \alpha \rho^\varepsilon. \end{cases} \quad (12)$$

The formal limit as $\varepsilon \rightarrow 0$ leads to the usual Keller–Segel model with nonlinear diffusion (see [5])

$$\begin{cases} \rho_t^0 + \nabla \cdot (\rho^0 \nabla (c^0 - g(\rho^0))) = 0 \\ \Delta c^0 + \alpha \rho^0 = 0. \end{cases} \quad (13)$$

2.3. Third scaling: Diffusive scaling with small reaction rates. Starting once again by (6), we consider the case $\sigma = d = 1$. We consider ε -depending reaction coefficients α and β , namely we require $\alpha = \varepsilon \tilde{\alpha}$ and $\beta = \varepsilon \tilde{\beta}$ for fixed $\tilde{\alpha}, \tilde{\beta} > 0$. We then perform the diffusive scaling

$$\begin{aligned} \tau &= \frac{t}{\varepsilon^2}, \quad y = \frac{x}{\varepsilon}, \\ v^\varepsilon(x, t) &= \frac{1}{\varepsilon} v(x, \tau), \quad \rho^\varepsilon(x, t) = \rho(x, \tau), \quad c^\varepsilon(x, t) = c(x, \tau). \end{aligned} \quad (14)$$

This leads to the rescaled system

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon v^\varepsilon) = 0 \\ \varepsilon^2 [\partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon] + \nabla g(\rho^\varepsilon) = \nabla c^\varepsilon - v^\varepsilon \\ \partial_t c^\varepsilon = \Delta c^\varepsilon + \tilde{\alpha} \rho^\varepsilon - \tilde{\beta} c^\varepsilon. \end{cases} \quad (15)$$

Therefore, the formal limit as $\varepsilon \rightarrow 0$ is given in this case by the following fully parabolic model (we drop the $\tilde{\sim}$ symbol for simplicity)

$$\begin{cases} \rho_t^0 + \nabla \cdot (\rho^0 \nabla (c^0 - g(\rho^0))) = 0 \\ c_t^0 = \Delta c^0 + \alpha \rho^0 - \beta c^0. \end{cases} \quad (16)$$

Remark 2. From the hypotheses (A3) and the scalings (7), (11), (14), it follows that the sequences $\{\rho^\varepsilon\}$, $\{\varepsilon \rho^\varepsilon v^\varepsilon\}$ are uniformly bounded in $L^\infty(\mathbb{T}^2 \times [0, +\infty))$ with respect to ε .

3. Estimates. In this section we provide suitable estimates on the solutions of the three rescaled models (8), (12) and (15). For future use we define

$$P(\rho) := \int_0^\rho g(n)dn = \frac{1}{\gamma+1}\rho^{\gamma+1}. \quad (17)$$

3.1. First scaling. We have the following standard energy estimate for the rescaled system (8).

Proposition 1. *The following identity is satisfied for any $t \in [0, T]$, by any solution $(\rho^\varepsilon, v^\varepsilon, c^\varepsilon)$ to (8):*

$$\begin{aligned} & \int_{\mathbb{T}^2} \left[\frac{\varepsilon^2}{2} \rho^\varepsilon(x, t) |v^\varepsilon(x, t)|^2 + P(\rho^\varepsilon(x, t)) \right] dx + \frac{1}{2} \int_0^t \int_{\mathbb{T}^2} \rho^\varepsilon(x, s) |v^\varepsilon(x, s)|^2 dx ds \\ & \leq \int_{\mathbb{T}^2} \left[\frac{\varepsilon^2}{2} \rho_0^\varepsilon(x) |v_0^\varepsilon(x)|^2 + P(\rho_0^\varepsilon(x)) \right] dx + \left(\tilde{K}t + \varepsilon \int_{\mathbb{T}^2} \frac{c_0^\varepsilon(x)}{2} dx \right). \end{aligned} \quad (18)$$

Proof. By multiplying second equation in (8) by $\rho^\varepsilon v^\varepsilon$ by using the first equation in (8) and by integration by parts it follows that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{T}^2} \left[\frac{\varepsilon^2}{2} \rho^\varepsilon(x, t) |v^\varepsilon(x, t)|^2 + P(\rho^\varepsilon(x, t)) \right] dx + \int_{\mathbb{T}^2} \rho^\varepsilon(x, s) |v^\varepsilon(x, s)|^2 dx \\ & = \int_{\mathbb{T}^2} \rho^\varepsilon(x, s) v^\varepsilon(x, s) \nabla c^\varepsilon(x, t) dx \leq \frac{1}{2} \int_{\mathbb{T}^2} \rho^\varepsilon(x, s) |v^\varepsilon(x, s)|^2 dx \\ & \quad + \frac{1}{2} \|\rho^\varepsilon\|_\infty \int_{\mathbb{T}^2} |\nabla c^\varepsilon(x, t)|^2 dx. \end{aligned} \quad (19)$$

Now, by multiplying the third equation of (8) by c^ε we get, for any $\delta > 0$,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}^2} \frac{\varepsilon}{2} |c^\varepsilon(x, t)|^2 dx & = - \int_{\mathbb{T}^2} |\nabla c^\varepsilon(x, t)|^2 dx + \alpha \|\rho^\varepsilon\|_\infty \left(\frac{|\mathbb{T}^2|^2}{4\delta} + \delta \int_{\mathbb{T}^2} |c^\varepsilon(x, t)|^2 dx \right) \\ & \quad - \beta \int_{\mathbb{T}^2} |c^\varepsilon(x, t)|^2 dx. \end{aligned} \quad (20)$$

By choosing $\delta < \frac{\beta}{2}$, by integrating in time we obtain, for fixed constant \tilde{K} , independent on ε , that c^ε satisfies the following inequality

$$\begin{aligned} & \int_{\mathbb{T}^2} \frac{\varepsilon}{2} |c^\varepsilon(x, t)|^2 dx + \frac{\beta}{2} \int_0^t \int_{\mathbb{T}^2} |c^\varepsilon(x, t)|^2 dx ds \\ & + \int_0^t \int_{\mathbb{T}^2} |\nabla c^\varepsilon(x, t)|^2 dx ds \leq \tilde{K}t + \varepsilon \int_{\mathbb{T}^2} \frac{|c_0^\varepsilon(x)|^2}{2} dx. \end{aligned} \quad (21)$$

The estimate (18) follows now by using together (19) with (21) and by taking into account the hypothesis (A3). \square

Corollary 1. *Let $(\rho^\varepsilon, v^\varepsilon, c^\varepsilon)$ be a solution to (8) satisfying assumption 1 and with initial datum $(\rho_0^\varepsilon, v_0^\varepsilon, c_0^\varepsilon)$ satisfying*

$$\int_{\mathbb{T}^2} \left[\frac{\varepsilon^2}{2} \rho_0^\varepsilon |v_0^\varepsilon|^2 dx + (\rho_0^\varepsilon)^{\gamma+1} + \varepsilon |c_0^\varepsilon|^2 \right] dx \quad \text{uniformly bounded w.r.t. } \varepsilon \ll 1. \quad (22)$$

Then, for all $T > 0$,

$$\varepsilon\sqrt{\rho^\varepsilon}v^\varepsilon \quad \text{is uniformly bounded in } L^\infty([0, T], L^p(\mathbb{T}^2)), \text{ for all } p \geq 1, \quad (23)$$

$$\rho^\varepsilon \quad \text{is uniformly bounded in } L^\infty([0, T], L^p(\mathbb{T}^2)), \text{ for all } p \geq 1, \quad (24)$$

$$\sqrt{\rho^\varepsilon}v^\varepsilon \quad \text{is uniformly bounded in } L^2([0, T] \times \mathbb{T}^2), \quad (25)$$

$$\sqrt{\varepsilon}c^\varepsilon \quad \text{is uniformly bounded in } L^\infty([0, T], L^2(\mathbb{T}^2)), \quad (26)$$

$$c^\varepsilon \quad \text{is uniformly bounded in } L^2([0, T], H^1(\mathbb{T}^2)). \quad (27)$$

Proof. (23) and (24) are a consequence of the assumption 1, while (25) follows from the inequality (18). Finally (26), (27) follow from (21). \square

Remark 3. In the above result a ‘closed’ energy structure is hidden in the computation. In fact one can easily prove that

$$\begin{aligned} & \frac{d}{dt} \left[\frac{\varepsilon^2}{2} \int \rho |v|^2 dx + \int P(\rho) dx + \frac{\beta}{\alpha} \int c^2 dx + \frac{1}{2} \int |\nabla c|^2 dx - \int \rho c dx \right] \\ &= - \int \rho |v|^2 dx. \end{aligned}$$

This computation would lead to the same estimates proven in Corollary 1. Indeed, the assumption 1 on ρ implies that the term $\frac{\beta}{\alpha} \int c^2 dx - \int \rho c$ is uniformly bounded from below.

3.2. Second scaling. We consider the following energy for the solution to (12)

$$E_\varepsilon(t) = \int_{\mathbb{R}^2} \left[\frac{\varepsilon^2}{2} \rho^\varepsilon |v^\varepsilon|^2 + P(\rho^\varepsilon) - \frac{1}{2} \rho^\varepsilon c^\varepsilon \right] dx, \quad (28)$$

where P is given by (17). For simplicity we will take $\alpha = 1$. We have the following estimate.

Proposition 2. *The following equality is valid for any solution $(\rho^\varepsilon, v^\varepsilon, c^\varepsilon)$ to (12):*

$$E_\varepsilon(t) + \int_0^t \int_{\mathbb{T}^2} \rho^\varepsilon(x, s) |v^\varepsilon(x, s)|^2 dx ds = E_\varepsilon(0) \quad (29)$$

Proof. By using the Poisson equation of the system (12) we easily have

$$\begin{aligned} \frac{d}{dt} E_\varepsilon(t) &= - \int_{\mathbb{T}^2} \rho^\varepsilon |v^\varepsilon|^2 dx + \int_{\mathbb{T}^2} \rho^\varepsilon v^\varepsilon \cdot \nabla c^\varepsilon dx + \int_{\mathbb{T}^2} c^\varepsilon \Delta c_t^\varepsilon dx \\ &= - \int_{\mathbb{T}^2} \rho^\varepsilon |v^\varepsilon|^2 dx + \int_{\mathbb{T}^2} \rho^\varepsilon v^\varepsilon \cdot \nabla c^\varepsilon dx + \int_{\mathbb{T}^2} (c^\varepsilon \nabla \cdot (\rho^\varepsilon v^\varepsilon)) dx, \end{aligned}$$

and this implies

$$\frac{d}{dt} E_\varepsilon(t) = - \int_{\mathbb{T}^2} \rho^\varepsilon |v^\varepsilon|^2 dx.$$

Integration with respect to time yields (29). \square

In order to recover an estimate for ∇c^ε , let us introduce the following convex functional

$$J[\rho^\varepsilon] = \int_{\mathbb{T}^2} (P(\rho^\varepsilon(t)) - \rho^\varepsilon c^\varepsilon(t)) dx.$$

Now we proceed by estimating the functional $J[\rho^\varepsilon]$ from below, using the same strategy of [5]. Let us recall that if $c^\varepsilon \in W^{1,1}(\mathbb{T}^2)$, then the convex functional $J[\rho^\varepsilon]$ has a critical point ρ^* which is a solution of

$$g(\rho^*) - c^\varepsilon = \lambda \quad (30)$$

whenever $\rho^* > 0$ and null otherwise. Here λ is the Lagrange multiplier associated to the constraint given by the mass conservation $\int \rho^* = M$ and fixed by this condition. We refer to [5] and ([6], Proposition 5) for details. Therefore, we have

$$J[\rho^\varepsilon] \geq \int_{\mathbb{T}^2} (P(\rho^*) - \rho^* c^\varepsilon) dx = \int_{\{\rho^* > 0\}} (P(\rho^*) - \rho^* g(\rho^*) + \lambda \rho^*) dx.$$

By taking into account the Remark 1 we can introduce the corrective term R such that $g(\rho^*) = \kappa \log \rho^* + R(\rho^*)$, then we have

$$J[\rho^\varepsilon] \geq \int_{\mathbb{T}^2} (P(\rho^*) - \kappa \rho^* \log \rho^*) dx - \int_{\{\rho^* > 0\}} \rho^* R(\rho^*) dx + \lambda M. \quad (31)$$

Now, (30) implies $\kappa \log \rho^* + R(\rho^*) = \lambda + c^\varepsilon$, whenever $\rho^* > 0$ and thus

$$\int_{\{\rho^* > 0\}} \exp\left(\frac{R(\rho^*)}{\kappa}\right) \rho^* dx = e^{\lambda/\kappa} \int_{\{\rho^* > 0\}} \exp\left(\frac{c^\varepsilon}{\kappa}\right) dx,$$

so we have

$$\lambda = \kappa \log \left(\int_{\{\rho^* > 0\}} e^{R/\kappa} \rho^* dx \right) - \kappa \log \left(\int_{\{\rho^* > 0\}} e^{c^\varepsilon/\kappa} dx \right). \quad (32)$$

If we replace λ by its expression in the inequality (31), we conclude that

$$\begin{aligned} J[\rho^\varepsilon] &\geq \int_{\mathbb{T}^2} (P(\rho^*) - \kappa \rho^* \log \rho^*) dx - \int_{\{\rho^* > 0\}} \rho^* R(\rho^*) dx \\ &\quad + \kappa M \log \left(\int_{\{\rho^* > 0\}} e^{R/\kappa} \rho^* dx \right) - \kappa M \log \left(\int_{\{\rho^* > 0\}} e^{c^\varepsilon/\kappa} dx \right). \end{aligned} \quad (33)$$

By taking into account the Remark 1 we have that

$$\int_{\{\rho^* \geq \mathcal{U}\}} (P(\rho^*) - \kappa \rho^* \log \rho^*) dx \geq C.$$

On the other hand, we have

$$\int_{\{\rho^* < \mathcal{U}\}} (P(\rho^*) - \kappa \rho^* \log \rho^*) dx \geq - \left(\sup_{[0, \mathcal{U})} (P - \kappa \rho \log \rho)^- \right) |\mathbb{T}^2|.$$

Therefore,

$$\int_{\mathbb{T}^2} (P(\rho^*) - \rho^* \log \rho^*) dx$$

is uniformly bounded from below. Now, the Jensen inequality for the probability density ρ^*/M over the set where $\rho^* > 0$, gives us that

$$\exp \int_{\{\rho^* > 0\}} \left(\frac{R(\rho^*)}{\kappa} \frac{\rho^*}{M} dx \right) \leq \int_{\{\rho^* > 0\}} e^{R/\kappa} \frac{\rho^*}{M} dx,$$

and thus

$$\kappa M \log \left(\int_{\{\rho^* > 0\}} e^{R/\kappa} \frac{\rho^*}{M} dx \right) - \int_{\{\rho^* > 0\}} \rho^* R(\rho^*) dx \geq 0.$$

Finally, we recall and use the Trudinger - Moser inequality [40, 9, 23, 43].

Theorem 3.1. *Assume that $\Omega \subset \mathbb{R}^2$ is a C^2 , bounded, connected domain. It exists a constant C_Ω , such that for all $h \in H^1$ with $\int_\Omega h = 0$ we have*

$$\int_\Omega \exp(|h|) dx \leq C_\Omega \exp\left(\frac{1}{8\pi} \int_\Omega |\nabla h|^2 dx\right).$$

By applying the previous theorem to c^ε/κ we obtain

$$\int_{\{\rho^* > 0\}} e^{c^\varepsilon/\kappa} dx \leq \int_{\mathbb{T}^2} e^{c^\varepsilon/\kappa} dx \leq \exp\left(\frac{1}{8\pi\kappa^2} \int_{\mathbb{T}^2} |\nabla c^\varepsilon|^2 dx\right)$$

and thus

$$-\kappa M \log\left(\int_{\{\rho^* > 0\}} e^{c^\varepsilon/\kappa} dx\right) \geq -\frac{M}{8\pi\kappa^2} \int_{\mathbb{T}^2} |\nabla c^\varepsilon|^2 dx.$$

So by (33) we have that

$$J[\rho^\varepsilon] \geq C - \frac{M}{8\pi\kappa^2} \int_{\mathbb{T}^2} |\nabla c^\varepsilon|^2 dx \quad (34)$$

Proposition 3. *Assume $(\rho^\varepsilon, v^\varepsilon, c^\varepsilon)$ be a solution to (12) satisfying assumption 1 then*

$$\int_{\mathbb{T}^2} |\nabla c^\varepsilon|^2 dx \quad \text{is uniformly bounded.} \quad (35)$$

Proof. We can rewrite (29) as

$$\begin{aligned} E_\varepsilon(0) &= \int_{\mathbb{T}^2} \frac{\varepsilon^2}{2} \rho^\varepsilon |v^\varepsilon|^2 dx + \int_{\mathbb{T}^2} J[\rho^\varepsilon] + \frac{1}{2} \int_{\mathbb{T}^2} |\nabla c^\varepsilon(t)|^2 dx \\ &\quad + \int_0^t \int_{\mathbb{T}^2} \rho^\varepsilon(x, s) |v^\varepsilon(x, s)|^2 dx ds. \end{aligned} \quad (36)$$

Combining (36) with (34) we get that

$$\begin{aligned} E_\varepsilon(0) &\geq \int_{\mathbb{T}^2} \frac{\varepsilon^2}{2} \rho^\varepsilon |v^\varepsilon|^2 dx + \int_0^t \int_{\mathbb{T}^2} \rho^\varepsilon(x, s) |v^\varepsilon(x, s)|^2 dx ds \\ &\quad + C|\mathbb{T}^2| + \frac{1}{2} \left(1 - \frac{M}{4\pi\kappa}\right) \int_{\mathbb{T}^2} |\nabla c^\varepsilon(t)|^2 dx. \end{aligned} \quad (37)$$

Finally, Remark 1 implies $\kappa > \kappa^*$, i.e. $(1 - \frac{M}{4\pi\kappa}) > 0$ and thus

$$\int_{\mathbb{T}^2} |\nabla c^\varepsilon|^2 dx$$

is uniformly bounded. \square

Corollary 2. *Let $(\rho^\varepsilon, v^\varepsilon, c^\varepsilon)$ be a solution to (12) satisfying assumption 1 and with initial datum $(\rho_0^\varepsilon, v_0^\varepsilon, c_0^\varepsilon)$ satisfying*

$$\int_{\mathbb{T}^2} \left[\frac{\varepsilon^2}{2} \rho_0^\varepsilon |v_0^\varepsilon|^2 dx + (\rho_0^\varepsilon)^{\gamma+1} - \frac{1}{2} \rho_0^\varepsilon c_0^\varepsilon \right] dx \quad \text{uniformly bounded w.r.t. } \varepsilon \ll 1. \quad (38)$$

Then, for all $T > 0$,

$$\varepsilon \sqrt{\rho^\varepsilon} v^\varepsilon \quad \text{is uniformly bounded in } L^\infty([0, T], L^p(\mathbb{T}^2)), \text{ for all } p \geq 1, \quad (39)$$

$$\rho^\varepsilon \quad \text{is uniformly bounded in } L^\infty([0, T], L^p(\mathbb{T}^2)), \text{ for all } p \geq 1, \quad (40)$$

$$\sqrt{\rho^\varepsilon} v^\varepsilon \quad \text{is uniformly bounded in } L^2([0, T] \times \mathbb{T}^2), \quad (41)$$

$$c^\varepsilon \quad \text{is uniformly bounded in } L^\infty([0, T], H^1(\mathbb{T}^2)). \quad (42)$$

Proof. (42) follows from Proposition 3 and by taking into account that we are in a periodic domain. (39) and (40) are a consequence of the assumption 1, while (41) is a consequence of (29) and (35). \square

3.3. Third scaling. With the same procedure as in the Proposition 1 we are able to prove the following proposition.

Proposition 4. *Let $(\rho^\varepsilon, v^\varepsilon, c^\varepsilon)$ be a solution to (15) satisfying assumption 1 and with initial datum $(\rho_0^\varepsilon, v_0^\varepsilon, c_0^\varepsilon)$ satisfying*

$$\int_{\mathbb{T}^2} \left[\frac{\varepsilon^2}{2} \rho_0^\varepsilon |v_0^\varepsilon|^2 dx + (\rho_0^\varepsilon)^{\gamma+1} + |c_0^\varepsilon|^2 \right] dx \quad \text{uniformly bounded w.r.t. } \varepsilon \ll 1. \quad (43)$$

Then, for all $T > 0$,

$$\varepsilon \sqrt{\rho^\varepsilon} v^\varepsilon \quad \text{is uniformly bounded in } L^\infty([0, T], L^p(\mathbb{T}^2)), \text{ for all } p \geq 1, \quad (44)$$

$$\rho^\varepsilon \quad \text{is uniformly bounded in } L^\infty([0, T], L^p(\mathbb{T}^2)), \text{ for all } p \geq 1, \quad (45)$$

$$\sqrt{\rho^\varepsilon} v^\varepsilon \quad \text{is uniformly bounded in } L^2([0, T] \times \mathbb{T}^2), \quad (46)$$

$$c^\varepsilon \quad \text{is uniformly bounded in } L^\infty([0, T], L^2(\mathbb{T}^2)) \cap L^2([0, T], H^1(\mathbb{T}^2)). \quad (47)$$

4. Strong convergence. This section is devoted to the study of the relaxation of the systems (8), (12), (15) towards their formal limit (9), (13), (16), respectively. As a consequence of the Corollary 1 and the Propositions 3, 4 we have that, extracting if necessary a subsequence,

$$\nabla c^\varepsilon \rightharpoonup \nabla c^0 \quad \text{weakly in } L^2([0, T] \times \mathbb{T}^2), \text{ as } \varepsilon \downarrow 0.$$

This convergence for c^ε is enough to pass into the limit in (8), (12), (15) and to get, in the sense of distribution, (9), (13), (16), respectively, provided that ρ^ε converges in a strong topology. In fact by the Remark 2, we know that $\rho^\varepsilon \rightarrow \rho^0$ in L^∞ *-weakly, while by (24), (40), (45) we have $\rho^\varepsilon \rightharpoonup \rho^0$ weakly in L^p , for any $p > 1$. These convergence are clearly too weak to pass into the limit in the nonlinear terms of (8), (12), (15). So, in this section we will investigate the strong convergence of the approximating sequence ρ^ε . The analysis of this convergence reduces to the analysis of the convergence of quadratic forms with constant coefficients via the classical compensated compactness technique due to Tartar [46, 47] and Murat [41] (see Dacorogna [10]). As we will see later on, these techniques will apply in the same way to the three scalings (7), (11), (14), so we will discuss them together. Let us recall the following theorem

Theorem 4.1. (Tartar's Compensated compactness)

Let us consider

1. a bounded open set $\Omega \subset \mathbb{R}^n$;
2. a sequence $\{l^\nu\}_{\nu=1}^\infty$, $l^\nu : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$;
3. a symmetric matrix $\Theta : \mathbb{R}^m \rightarrow \mathbb{R}^m$;
4. constants $a_{jk}^i \in \mathbb{R}$, $i = 1, \dots, q$, $j = 1, \dots, m$, $k = 1, \dots, n$.

Let us define

$$f(\alpha) = \langle \Theta \alpha, \alpha \rangle, \quad \text{for all } \alpha \in \mathbb{R}^m;$$

$$\Lambda = \left\{ \lambda \in \mathbb{R}^m : \exists \eta \in \mathbb{R}^n \setminus \{0\}, \sum_{j,k} a_{jk}^i \lambda_j \eta_k = 0, i = 1, \dots, q \right\}.$$

Assume that

- (a) there exists $\tilde{l} \in L^2(\Omega)$ such that $l^\nu \rightharpoonup \tilde{l}$ in $L^2(\Omega)$ as $\nu \uparrow \infty$;
- (b) $\mathcal{A}^i l^\nu = \sum_{j,k} a_{jk}^i \frac{\partial l_j^\nu}{\partial x_k}$, $i = 1, \dots, q$ are relatively compact in $H_{loc}^{-1}(\Omega)$;
- (c) $f|_\Lambda \equiv 0$;
- (d) there exists $\tilde{f} \in \mathbb{R}$ such that $f(l) \rightharpoonup \tilde{f}$ in the sense of measures $\mathcal{M}(\Omega)$.

Then we have $\tilde{f} = f(\tilde{l})$.

4.1. Weak convergence of $\rho^\varepsilon P(\rho^\varepsilon)$. First of all we start by studying the weak convergence of $\rho^\varepsilon P(\rho^\varepsilon)$. Our goal will be to prove that

$$\rho^\varepsilon P(\rho^\varepsilon) \rightharpoonup \rho^0 P(\rho^0),$$

where ρ^0 is the weak limit of ρ^ε . To this end we are going to apply the Theorem 4.1 in the same spirit of [39]. In order to fit the into the hypotheses of the Theorem 4.1 we rewrite the first two equations of the systems (8), (12), (15), as

$$\begin{cases} \rho_t^\varepsilon + m_x^\varepsilon + n_y^\varepsilon = 0 \\ \varepsilon^2 m_t^\varepsilon + \left(\varepsilon^2 \frac{(m^\varepsilon)^2}{\rho^\varepsilon} + \gamma P(\rho^\varepsilon) \right)_x + \left(\varepsilon^2 \frac{m^\varepsilon n^\varepsilon}{\rho^\varepsilon} \right)_y = \rho^\varepsilon c_x^\varepsilon - m^\varepsilon \\ \varepsilon^2 n_t^\varepsilon + \left(\varepsilon^2 \frac{m^\varepsilon n^\varepsilon}{\rho^\varepsilon} \right)_x + \left(\varepsilon^2 \frac{(n^\varepsilon)^2}{\rho^\varepsilon} + \gamma P(\rho^\varepsilon) \right)_y = \rho^\varepsilon c_y^\varepsilon - n^\varepsilon. \end{cases} \quad (48)$$

where

$$v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon) \quad \rho^\varepsilon v^\varepsilon = (\rho^\varepsilon v_1, \rho^\varepsilon v_2) = (m^\varepsilon, n^\varepsilon). \quad (49)$$

It will be useful to rewrite (48) in the following way

$$\begin{aligned} \rho_t^\varepsilon + m_x^\varepsilon + n_y^\varepsilon &= 0 \\ \gamma P(\rho^\varepsilon)_x &= -\varepsilon^2 m_t^\varepsilon - \varepsilon^2 \left(\frac{(m^\varepsilon)^2}{\rho^\varepsilon} \right)_x - \varepsilon^2 \left(\frac{m^\varepsilon n^\varepsilon}{\rho^\varepsilon} \right)_y + \rho^\varepsilon c_x^\varepsilon - m^\varepsilon \\ \gamma P(\rho^\varepsilon)_y &= -\varepsilon^2 n_t^\varepsilon - \varepsilon^2 \left(\frac{m^\varepsilon n^\varepsilon}{\rho^\varepsilon} \right)_x - \varepsilon^2 \left(\frac{(n^\varepsilon)^2}{\rho^\varepsilon} \right)_y + \rho^\varepsilon c_y^\varepsilon - n^\varepsilon. \end{aligned} \quad (50)$$

By using (25), (41), (46), (35), and the assumption (A3) we get that $\rho^\varepsilon v^\varepsilon$, $\rho^\varepsilon \nabla c^\varepsilon \in L^2([0, T] \times \mathbb{T}^2)$. In fact

$$\|\rho^\varepsilon v^\varepsilon\|_{L^2([0, T] \times \mathbb{T}^2)} \leq \|\sqrt{\rho^\varepsilon}\|_\infty \|\sqrt{\rho^\varepsilon} v^\varepsilon\|_{L^2([0, T] \times \mathbb{T}^2)} \quad (51)$$

$$\|\rho^\varepsilon \nabla c^\varepsilon\|_{L^2([0, T] \times \mathbb{T}^2)} \leq \|\rho^\varepsilon\|_\infty \|\nabla c^\varepsilon\|_{L^2([0, T] \times \mathbb{T}^2)} \quad (52)$$

Moreover, by taking into account the assumptions (A2) and (A3) and (25), (41), (46) we have that $\varepsilon^2 \left(\frac{(m^\varepsilon)^2}{\rho^\varepsilon} \right)_x$, is relatively compact in $H_{loc}^{-1}([0, T] \times \mathbb{T}^2)$. In fact let us consider ω relatively compact in $[0, T] \times \mathbb{T}^2$, then by taking into account (A2), (A3) and the Remark 2 we have,

$$\begin{aligned} \left\| \varepsilon^2 \left(\frac{(m^\varepsilon)^2}{\rho^\varepsilon} \right)_x \right\|_{H^{-1}(\omega)} &\leq \sup_{\|\phi\|_{H_0^1(\omega)}=1} \left| \left\langle \varepsilon^2 \left(\frac{(m^\varepsilon)^2}{\rho^\varepsilon} \right)_x, \phi \right\rangle \right| \\ &\leq \varepsilon \|\rho^\varepsilon v^\varepsilon\|_\infty \frac{1}{\sqrt{k}} \|\sqrt{\rho^\varepsilon} v^\varepsilon\|_{L^2(\omega)} \end{aligned} \quad (53)$$

In a similar way it can be proved that the terms

$$\varepsilon^2 \left(\frac{m^\varepsilon n^\varepsilon}{\rho^\varepsilon} \right)_y, \quad \varepsilon^2 \left(\frac{m^\varepsilon n^\varepsilon}{\rho^\varepsilon} \right)_x, \quad \varepsilon^2 \left(\frac{(n^\varepsilon)^2}{\rho^\varepsilon} \right)_y, \quad \varepsilon^2 (\rho^\varepsilon v^\varepsilon)_t$$

are relatively compact in $H^{-1}([0, T] \times \mathbb{T}^2)$. Now, (51)–(53), imply that

$$\begin{pmatrix} \rho_i^\varepsilon + m_x^\varepsilon + n_y^\varepsilon \\ P(\rho^\varepsilon)_x \\ P(\rho^\varepsilon)_y \end{pmatrix} \text{ is relatively compact in } (H_{loc}^{-1})^3. \quad (54)$$

In order to fit into the framework of the Theorem 4.1 we set $x_1 = x$, $x_2 = y$, $x_3 = t$, $l^\varepsilon = (m^\varepsilon, n^\varepsilon, \rho^\varepsilon, P(\rho^\varepsilon))$, hence $m = 4$. In our case the differential constraints are $q = 3$. So we can define the matrices $\mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3 \in \mathcal{M}_{4 \times 3}$, where $\mathcal{A}^i = \{a_{jk}^i\}$, $i = 1, 2, 3$, $j = 1, \dots, 4$, $k = 1, 2, 3$ as follows:

$$\mathcal{A}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathcal{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathcal{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The characteristic manifold Λ is then given by

$$\Lambda = \{\lambda \in \mathbb{R}^4 \mid \exists \xi \in \mathbb{R}^3 \setminus \{0\}, B(\xi, \lambda) = 0\}$$

where

$$B(\xi, \lambda) = \begin{pmatrix} \lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3 \\ \lambda_4 \xi_1 \\ \lambda_4 \xi_2 \end{pmatrix}.$$

Therefore

$$\Lambda = \left\{ \lambda \in \mathbb{R}^4 \mid \det \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & 0 & 0 \\ 0 & \lambda_4 & 0 \end{pmatrix} \right\} = \{\lambda \in \mathbb{R}^4 \mid \lambda_3 \lambda_4 = 0\}.$$

If we define

$$M = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{M}_{4 \times 4},$$

then $f(\lambda) = \lambda^T M \lambda = \lambda_3 \lambda_4$ and, of course $f|_\Lambda \equiv 0$. Now, by applying the Theorem 4.1 we have $f(l^\varepsilon) \rightharpoonup f(\tilde{l})$, and in our case this means

$$\rho^\varepsilon P(\rho^\varepsilon) \rightharpoonup \rho^0 P^0,$$

where $P^0 = w - \lim P(\rho^\varepsilon)$.

4.2. Strong convergence of ρ^ε . In the previous section we proved that

$$\rho^\varepsilon P(\rho^\varepsilon) \rightharpoonup \rho^0 P^0.$$

Here we will be able to prove that

$$\rho^\varepsilon \rightarrow \rho^0 \text{ strongly in } L^p, p < +\infty.$$

At this point we can follow the methods of [36], [39]. First of all let us use Minty's argument ([33], [36]) to prove that $P^0 = P(\rho^0)$. Since the function P is monotone, for any $w \in L^\infty$ and φ test function, $\varphi > 0$, we have that

$$H(\varepsilon) \equiv \iint (P(\rho^\varepsilon) - P(w))(\rho^\varepsilon - w)\varphi dxdt \geq 0. \quad (55)$$

But, for $\varepsilon \downarrow 0$, we have that

$$\iint P(\rho^\varepsilon)\rho^\varepsilon\varphi dxdt \rightarrow \iint P^0\rho^0\varphi dxdt.$$

So from (55) we get that for $\varepsilon \downarrow 0$

$$H(\varepsilon) \rightarrow H \equiv \iint (P^0 - p(w))(\rho^0 - w)\varphi dxdt \geq 0.$$

If we choose $w = \rho^0 + \lambda z$, with $\lambda \leq 0$ and arbitrary $z \in L^\infty$, we have

$$\begin{aligned} G(\lambda, z) &\equiv \iint (P^0 - P(\rho^0 + \lambda z))z\varphi dxdt \\ &= \frac{1}{\lambda} \iint (P^0 - P(\rho^0 + \lambda z))\lambda z\varphi dxdt \leq 0 \end{aligned}$$

and for $\lambda \uparrow 0$, $G(0, z) \leq 0$ for any $z \in L^\infty$, then

$$G(0, z) = \iint (P^0 - P(\rho^0))z\varphi dxdt = 0,$$

and finally $P^0 = P(\rho^0)$.

Our next step now, is to prove the strong convergence for $\rho^\varepsilon \rightarrow \rho^0$ in L^p_{loc} . To this end we characterize the weak convergence by means of Young's probability measures (see [46],[13], [14], [15], [16]). Let us recall that if $\{u^\varepsilon\}$ is sequence converging to U in L^∞ *-weakly, we can associate to $\{u^\varepsilon\}$ a family $\{\nu_{(x,t)}(\lambda)\}$ of probability measures such that for any continuous function $F(\cdot)$

$$* - \lim_{\varepsilon \rightarrow 0} F(u^\varepsilon) = \int F(\lambda)\nu_{(x,t)}(d\lambda) \quad a.e.$$

If $\nu_{(x,t)} = \delta_{U(x,t)}$ then $u^\varepsilon \rightarrow U$ strongly in L^p_{loc} for any $p \in (1, +\infty)$ (see [10], Corollary 6.2). Let $\{\nu_{(x,t)}\}$ be the family of Young's probability measures associated to the sequence $\{\rho^\varepsilon\}$: since $\rho^\varepsilon \rightarrow \rho^0$ in L^∞ *-weakly, we can find a closed interval $[a, b]$, $0 \leq a \leq b$, such that $supp \nu_{(x,t)} \subseteq [a, b]$. Since $P(r) = r^\alpha$, $\alpha > 1$, we have three possibilities:

1. $P \in C^2(\mathbb{R} \setminus \{0\})$ and $P''(r) \uparrow +\infty$ for $r \downarrow 0$, if $1 < \alpha < 2$;
2. $P \in C^2(\mathbb{R})$ and $P''(0) = 1$, if $\alpha = 2$;
3. $P \in C^2(\mathbb{R})$ and $P''(0) = 0$, if $\alpha > 2$.

Let us assume that $1 < \alpha \leq 2$. Then we can write for any λ, λ_0

$$P(\lambda) - P(\lambda_0) = P'(\lambda_0)(\lambda - \lambda_0) + \frac{1}{2}P''(\lambda^*)(\lambda - \lambda_0)^2,$$

where λ^* belongs to the segment between λ and λ_0 . If we choose

$$\lambda_0 = \int_a^b \lambda \nu_{(x,t)}(d\lambda) = \rho_0,$$

since $P^0 = P(\rho^0)$

$$P(\lambda_0) = \int_a^b P(\lambda) \nu_{(x,t)}(d\lambda)$$

so that

$$\int_a^b \{P(\lambda) - P(\lambda_0)\} \nu_{(x,t)}(d\lambda) = 0.$$

On the other hand we also have

$$\int_a^b P'(\lambda_0)(\lambda - \lambda_0) \nu_{(x,t)}(d\lambda) = P'(\lambda_0) \left\{ \int_a^b \lambda \nu_{(x,t)}(d\lambda) - \lambda_0 \int_a^b \nu_{(x,t)}(d\lambda) \right\} = 0,$$

so we can conclude that

$$\int_a^b P''(\lambda^*) (\lambda - \lambda_0)^2 \nu_{(x,t)}(d\lambda) = 0.$$

Taking $E = \min_{\lambda \in [a,b]} P''(\lambda) > 0$, we get

$$E \int_a^b (\lambda - \lambda_0)^2 \nu_{(x,t)}(d\lambda) \leq 0,$$

namely

$$\int_a^b (\lambda - \lambda_0)^2 \nu_{(x,t)}(d\lambda) = 0,$$

and it follows $a = b$ and $\nu_{(x,t)} = \delta$, a point mass and so we finally get

$$\rho^\varepsilon \rightarrow \rho_0 \quad \text{strongly in } L_{loc}^p.$$

To conclude we remark that in the case $\alpha > 2$ this result can be obtained in the same way by using the function $-P^{-1}$.

Remark 4. The strong convergence result for ρ^ε obtained in this section is still valid in the case of linear diffusion, namely if we consider $g(\rho) = \log \rho$ and consequently $P(\rho) = \rho \log \rho - \rho$.

By using the estimates and the strong convergence of the sequence $\{\rho^\varepsilon\}$ obtained in the previous section we are able to prove the following main theorem.

Theorem 4.2. *Let $T > 0$ be arbitrary and let $(\rho^\varepsilon, v^\varepsilon, c^\varepsilon)$ be a family of solutions to the system (8) ((12) and (15) respectively) with initial data satisfying (22) ((38) and (43) respectively). Assume that the assumption 1 holds, then, there exist $\rho^0 \in L^\infty([0, T] \times \mathbb{T}^2)$ and $c^0 \in L^2([0, T], H^1(\mathbb{T}^2))$, such that, extracting if necessary a subsequence,*

$$\begin{aligned} \rho^\varepsilon &\longrightarrow \rho^0 && \text{strongly in } L^p([0, T] \times \mathbb{T}^2) \text{ for any } p < \infty \\ \nabla c^\varepsilon &\rightharpoonup \nabla c^0 && \text{weakly in } L^2([0, T] \times \mathbb{T}^2). \end{aligned}$$

Moreover the couple (ρ^0, c^0) , satisfies the system (9) ((13) and (16) respectively) in the sense of distributions.

5. Perturbation of constant states in the approximating system. In this section we deal with the rescaled system

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon v^\varepsilon) = 0 \\ \varepsilon^2 [\partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon] + \nabla g(\rho^\varepsilon) = \nabla c^\varepsilon - v^\varepsilon \\ \varepsilon \partial_t c^\varepsilon = \Delta c^\varepsilon + \alpha \rho^\varepsilon - \beta c^\varepsilon, \end{cases} \quad (56)$$

with $x \in \mathbb{T}^2$, $t \geq 0$, where \mathbb{T}^2 is the flat normalized two-dimensional torus. The system (56) is complemented with the 1-periodical initial data

$$\rho^\varepsilon(x, 0) = \rho_0^\varepsilon(x), \quad v^\varepsilon(x, 0) = v_0^\varepsilon(x), \quad c^\varepsilon(x, 0) = c_0^\varepsilon(x).$$

We shall consider small perturbations of the stationary state

$$(\rho, v, c) = (\tilde{\rho}, \tilde{v}, \tilde{c}), \quad \tilde{\rho} > 0, \quad \tilde{v} = 0, \quad \tilde{c} = \frac{\alpha}{\beta} \tilde{\rho} \quad (57)$$

and we prove the existence of classical solutions $(\rho^\varepsilon, v^\varepsilon, c^\varepsilon)$ such that the density ρ^ε stays away from zero, uniformly in ε , on an arbitrary time interval $[0, T]$ (see similar results in [30] and [11]). In order to perform this task, we shall use an

iterative method, namely we define recursively the sequence (ρ^n, v^n, c^n) as follows: $(\rho^0(x, t), v^0(x, t), c^0(x, t)) = (\rho_0^\varepsilon(x), v_0^\varepsilon(x), c_0^\varepsilon(x))$ and, for $n \geq 1$, (ρ^n, v^n, c^n) solves the linear system

$$\begin{cases} \partial_t \rho^n + v^{n-1} \cdot \nabla \rho^n + \rho^{n-1} \nabla \cdot u^n = 0 \\ \partial_t v^n + v^{n-1} \cdot \nabla v^n + \frac{g'(\rho^{n-1})}{\varepsilon^2} \nabla \rho^n = \frac{1}{\varepsilon^2} \nabla c^n - \frac{1}{\varepsilon^2} v^n \\ \partial_t c^n = \frac{1}{\varepsilon} \Delta c^n + \frac{\alpha}{\varepsilon} \rho^n - \frac{\beta}{\varepsilon} c^n. \end{cases} \quad (58)$$

From now on we shall drop the dependency on ε on the solutions (ρ, v, c) for the sake of clarity. Moreover, we shall use the following notation: the variables taken at the step $n-1$ will be denoted e. g. by $\rho^{n-1} = \hat{\rho}$; the variables taken at the step n will be denoted without any additional symbol, e. g. $\rho^n = \rho$; the deviation from the aforementioned constant stationary states will be denoted e. g. by $\bar{\rho} = \rho^n - \tilde{\rho}$ and $\underline{\rho} = \rho^{n-1} - \tilde{\rho}$.

The first two equations in system (58) can be easily viewed as a hyperbolic system in vectorial form. More precisely, let us define the 3-dimensional variable U as

$$U := (\rho, v^1, v^2),$$

where $v = (v^1, v^2)$. Let us denote

$$A_1(\hat{U}) := \begin{bmatrix} \hat{v}^1 & \hat{\rho} & 0 \\ \frac{g'(\hat{\rho})}{\varepsilon^2} & \hat{v}^1 & 0 \\ 0 & 0 & \hat{v}^1 \end{bmatrix}, \quad A_2(\hat{U}) := \begin{bmatrix} \hat{v}^2 & 0 & \hat{\rho} \\ 0 & \hat{v}^2 & 0 \\ \frac{g'(\hat{\rho})}{\varepsilon^2} & 0 & \hat{v}^2 \end{bmatrix},$$

$$B(U) := \frac{1}{\varepsilon^2} \begin{pmatrix} 0 \\ \partial_{x_1} c - v^1 \\ \partial_{x_2} c - v^2 \end{pmatrix}.$$

Then, with all these notations, the system (58) can be rephrased as follows:

$$\begin{cases} \partial_t U + A_1(\hat{U}) \partial_{x_1} U + A_2(\hat{U}) \partial_{x_2} U = B(U) \\ \partial_t c = \frac{1}{\varepsilon} \Delta c + \frac{\alpha}{\varepsilon} \rho - \frac{\beta}{\varepsilon} c. \end{cases} \quad (59)$$

The first line in (59) corresponds to a linear hyperbolic system which can be *symmetrized* by means of the matrix

$$S(\hat{U}) := \text{diag} \left(\frac{g'(\hat{\rho})}{\varepsilon^2}, \hat{\rho}, \hat{\rho} \right). \quad (60)$$

The matrix $S(\hat{U})$ is uniformly positive definite provided the variable $\hat{\rho}$ satisfies a condition of the form $0 < c \leq \hat{\rho} \leq C$ (we recall that $g'(\rho) = \gamma \rho^{\gamma-1}$ exhibits a singularity at zero in case of $\gamma < 1$). The two matrices $S(\hat{U})A_1(\hat{U})$ and $S(\hat{U})A_2(\hat{U})$ can be easily proven to be symmetric. We now rewrite system (59) in terms of the deviations \bar{U} and \bar{c} :

$$\begin{cases} \partial_t \bar{U} + A_1(\bar{U} + \underline{U}) \partial_{x_1} \bar{U} + A_2(\bar{U} + \underline{U}) \partial_{x_2} \bar{U} = B(\bar{U} + \underline{U}) = B(\bar{U}) \\ \partial_t \bar{c} = \frac{1}{\varepsilon} \Delta \bar{c} + \frac{\alpha}{\varepsilon} \bar{\rho} - \frac{\beta}{\varepsilon} \bar{c}. \end{cases} \quad (61)$$

We introduce the energy functional

$$\mathcal{E}(U, c) := \frac{1}{2} \int_{\mathbb{T}^2} [U^T S(\hat{U}) U + \lambda c^2] dx = \frac{1}{2} \int_{\mathbb{T}^2} \left[\frac{g'(\hat{\rho})}{\varepsilon^2} \rho^2 + \hat{\rho} |v|^2 + \lambda c^2 \right] dx,$$

where $\lambda > 0$ is a constant to be chosen later on. We have the following

Proposition 5. *Let $T > 0$. There exist constants $\varepsilon_0, \delta \in (0, 1)$, $K \in (0, \tilde{\rho}/2)$ such that, if*

$$\begin{aligned} \|\rho_0^\varepsilon - \tilde{\rho}\|_{H^4(\mathbb{T}^2)} + \varepsilon\|v_0^\varepsilon\|_{H^4(\mathbb{T}^2)} + \sqrt{\varepsilon}\|c_0^\varepsilon - \tilde{c}\|_{H^4(\mathbb{T}^2)} &\leq \delta \quad \text{and} \\ \sup_{0 \leq t \leq T} (\|\underline{\rho}(t)\|_{H^4(\mathbb{T}^2)} + \varepsilon\|\underline{v}(t)\|_{H^4(\mathbb{T}^2)} + \sqrt{\varepsilon}\|\underline{c}(t)\|_{H^4(\mathbb{T}^2)}) &\leq K, \end{aligned} \quad (62)$$

for all $\varepsilon \in (0, \varepsilon_0)$, then,

$$\sup_{0 \leq t \leq T} (\|\bar{\rho}(t)\|_{H^4(\mathbb{T}^2)} + \varepsilon\|\bar{v}(t)\|_{H^4(\mathbb{T}^2)} + \sqrt{\varepsilon}\|\bar{c}(t)\|_{H^4(\mathbb{T}^2)}) \leq K \quad (63)$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. During the proof of this proposition we shall often make use of the Sobolev inequality $\|f\|_{L^\infty(\mathbb{T}^2)} \leq C\|f\|_{H^2(\mathbb{T}^2)}$.

STEP 1. Due to the symmetry of the two matrices SA_1 and SA_2 , we can use integration by parts in the evolution of $\mathcal{E}(\bar{U}, \bar{c})$ as follows:

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(\bar{U}, \bar{c}) &= \frac{1}{2} \int_{\mathbb{T}^2} \bar{U}^T \left[S(\hat{U})A_1(\hat{U}) \right]_{x_1} \bar{U} dx + \frac{1}{2} \int_{\mathbb{T}^2} \bar{U}^T \left[S(\hat{U})A_2(\hat{U}) \right]_{x_2} \bar{U} dx \\ &\quad + \int_{\mathbb{T}^2} \bar{U}^T S(\hat{U})B(\bar{U}) dx - \frac{\lambda}{\varepsilon} \int_{\mathbb{T}^2} |\nabla \bar{c}|^2 dx + \frac{\lambda\alpha}{\varepsilon} \int_{\mathbb{T}^2} \bar{\rho} \bar{c} dx - \frac{\lambda\beta}{\varepsilon} \int_{\mathbb{T}^2} \bar{c}^2 dx. \end{aligned} \quad (64)$$

Due to the assumption (62) we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(\bar{U}, \bar{c}) &\leq C(K) (\|\nabla \underline{\rho}\|_{L^\infty} + \|\nabla \underline{v}\|_{L^\infty}) \frac{1}{2} \int_{\mathbb{T}^2} \left(\frac{\bar{\rho}^2}{\varepsilon^2} + \frac{|\bar{v}|^2}{\varepsilon^2} \right) dx \\ &\quad + \frac{\|\bar{\rho}\|_{L^\infty}}{\varepsilon^2} \int_{\mathbb{T}^2} \bar{v} \cdot \nabla \bar{c} dx - \frac{(\tilde{\rho} - K)}{\varepsilon^2} \int_{\mathbb{T}^2} |\bar{v}|^2 dx - \frac{\lambda}{\varepsilon} \int_{\mathbb{T}^2} |\nabla \bar{c}|^2 dx \\ &\quad + \frac{\lambda\alpha}{\varepsilon} \int_{\mathbb{T}^2} \bar{\rho} \bar{c} dx - \frac{\lambda\beta}{\varepsilon} \int_{\mathbb{T}^2} \bar{c}^2 dx \end{aligned} \quad (65)$$

for a function $C(K) > 0$ of the constant K such that C is continuous on $K \in [0, \tilde{\rho}/2]$. By choosing

$$\lambda = \frac{(\tilde{\rho} + K)^2}{\varepsilon(\tilde{\rho} - K)}$$

and $\varepsilon_0 < 1$, we can use once again (62) and find a constant $C_1 > 0$ such that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(\bar{U}, \bar{c}) &\leq KC(K) \frac{1}{2} \int_{\mathbb{T}^2} \left(\frac{\bar{\rho}^2}{\varepsilon^2} + \frac{|\bar{v}|^2}{\varepsilon^2} \right) dx \\ &\quad - \frac{(\tilde{\rho} - K)}{2\varepsilon^2} \int_{\mathbb{T}^2} |\bar{v}|^2 dx - \frac{(\tilde{\rho} + K)^2}{2(\tilde{\rho} - K)\varepsilon^2} \int_{\mathbb{T}^2} |\nabla \bar{c}|^2 dx \\ &\quad - \frac{(\tilde{\rho} + K)^2\beta}{2(\tilde{\rho} - K)\varepsilon} \int_{\mathbb{T}^2} \bar{c}^2 dx + \frac{(\tilde{\rho} + K)^2\alpha^2}{2(\tilde{\rho} - K)\varepsilon\beta} \int_{\mathbb{T}^2} \bar{\rho}^2 dx. \end{aligned}$$

We now choose the constant K such that $KC(K) < \frac{1}{2}(\tilde{\rho} - K)$. By using the coercivity property

$$\mathcal{E}(U, c) \geq c(K) \int_{\mathbb{T}^2} \left[\frac{\rho^2}{\varepsilon^2} + |v|^2 + \frac{c^2}{\varepsilon} \right] dx, \quad (66)$$

which holds for a certain $c(K) > 0$, due to Gronwall inequality we easily obtain

$$\begin{aligned} \mathcal{E}(\bar{U}(t), \bar{c}(t)) + \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} [|\bar{v}(\tau)|^2 dx + |\nabla \bar{c}(\tau)|^2] dx d\tau \\ + \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^2} \bar{c}(\tau)^2 dx d\tau \leq A\mathcal{E}(\bar{U}(0), \bar{c}(0))e^{Bt} \end{aligned}$$

for certain constants $A, B > 0$ depending only on K and ε_0 . The above implies in particular

$$\sup_{0 \leq t \leq T} \int_{\mathbb{T}^2} [\bar{\rho}(t)^2 + \varepsilon^2 |\bar{v}(t)|^2 + \varepsilon \bar{c}(t)^2] dx dt \leq C(K) \delta e^{BT},$$

for a certain $C(K)$ depending on K . Therefore, by choosing δ small enough such that $C(K) \delta e^{BT} \leq K^2$ we obtain

$$\sup_{0 \leq t \leq T} (\|\bar{\rho}(t)\|_{L^2(\mathbb{T}^2)} + \varepsilon \|\bar{v}(t)\|_{L^2(\mathbb{T}^2)} + \sqrt{\varepsilon} \|\bar{c}(t)\|_{L^2(\mathbb{T}^2)}) \leq K.$$

STEP 2. We now perform the energy estimate of the space derivatives of (\bar{U}, \bar{c}) . For $j = 1, 2$ we denote the derivative with respect to x_j by the subscript $\rho_j = \partial_{x_j} \rho$. The system satisfied by (\bar{U}_j, \bar{c}_j) is

$$\begin{cases} \partial_t \bar{U}_j + A_1(\hat{U}) \partial_{x_1} \bar{U}_j + A_2(\hat{U}) \partial_{x_2} \bar{U}_j = B(\bar{U})_j - A_1(\hat{U})_j \bar{U}_1 - A_2(\hat{U})_j \bar{U}_2 \\ \partial_t \bar{c}_j = \frac{1}{\varepsilon} \Delta \bar{c}_j + \frac{\alpha}{\varepsilon} \bar{\rho}_j - \frac{\beta}{\varepsilon} \bar{c}_j. \end{cases} \quad (67)$$

The evaluation of the energy

$$\mathcal{E}(\bar{U}_j, \bar{c}_j) = \frac{1}{2} \int_{\mathbb{T}^2} [\bar{U}_j^T S(\hat{U}) \bar{U}_j + \lambda \bar{c}_j^2] dx$$

in a similar way as in (64) yields

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\bar{U}_j, \bar{c}_j) &= \frac{1}{2} \int_{\mathbb{T}^2} \bar{U}_j^T [S(\hat{U}) A_1(\hat{U})]_{x_1} \bar{U}_j dx + \frac{1}{2} \int_{\mathbb{T}^2} \bar{U}_j^T [S(\hat{U}) A_2(\hat{U})]_{x_2} \bar{U}_j dx \\ &+ \int_{\mathbb{T}^2} \bar{U}_j^T S(\hat{U}) B(\bar{U})_j dx - \frac{\lambda}{\varepsilon} \int_{\mathbb{T}^2} |\nabla \bar{c}_j|^2 dx \\ &+ \frac{\lambda \alpha}{\varepsilon} \int_{\mathbb{T}^2} \bar{\rho}_j \bar{c}_j dx - \frac{\lambda \beta}{\varepsilon} \int_{\mathbb{T}^2} \bar{c}_j^2 dx - \int_{\mathbb{T}^2} \bar{U}_j^T S(\hat{U}) A_1(\hat{U})_j \bar{U}_1 dx \\ &- \int_{\mathbb{T}^2} \bar{U}_j^T S(\hat{U}) A_2(\hat{U})_j \bar{U}_2 dx. \end{aligned}$$

Assumption (62) allows for the estimate of the first two terms above as in (64), as well as for the estimate of the last two terms in a similar fashion. The result is the following estimate

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\bar{U}_j, \bar{c}_j) &\leq \tilde{C}(K) (\|\nabla \underline{\rho}\|_{L^\infty} + \|\nabla \underline{v}\|_{L^\infty}) \frac{1}{2} \int_{\mathbb{T}^2} \left(\frac{\bar{\rho}_j^2}{\varepsilon^2} + \frac{|\bar{v}_j|^2}{\varepsilon^2} \right) dx \\ &+ \frac{\|\hat{\rho}\|_{L^\infty}}{\varepsilon^2} \int_{\mathbb{T}^2} \bar{v}_j \cdot \nabla \bar{c}_j dx - \frac{(\tilde{\rho} - K)}{\varepsilon^2} \int_{\mathbb{T}^2} |\bar{v}_j|^2 dx \\ &- \frac{\lambda}{\varepsilon} \int_{\mathbb{T}^2} |\nabla \bar{c}_j|^2 dx + \frac{\lambda \alpha}{\varepsilon} \int_{\mathbb{T}^2} \bar{\rho}_j \bar{c}_j dx - \frac{\lambda \beta}{\varepsilon} \int_{\mathbb{T}^2} \bar{c}_j^2 dx, \end{aligned}$$

which is the equivalent of the estimate (65) where (\bar{U}, \bar{c}) are replaced by their first derivatives. Therefore, we can easily conclude as before

$$\sup_{0 \leq t \leq T} (\|\nabla \bar{\rho}(t)\|_{L^2(\mathbb{T}^2)} + \varepsilon \|\nabla \bar{v}(t)\|_{L^2(\mathbb{T}^2)} + \sqrt{\varepsilon} \|\nabla \bar{c}(t)\|_{L^2(\mathbb{T}^2)}) \leq K.$$

STEP 3. The second space derivatives of (\bar{U}, \bar{c}) satisfy the system

$$\begin{cases} \partial_t \bar{U}_{ij} + A_1(\hat{U}) \partial_{x_1} \bar{U}_{ij} + A_2(\hat{U}) \partial_{x_2} \bar{U}_{ij} = B(\bar{U})_{ij} - A_1(\hat{U})_i \bar{U}_{1j} - A_2(\hat{U})_i \bar{U}_{2j} \\ \quad - A_1(\hat{U})_{ij} \bar{U}_1 - A_2(\hat{U})_{ij} \bar{U}_2 \\ \quad - A_1(\hat{U})_j \bar{U}_{1j} - A_2(\hat{U})_j \bar{U}_{2j} \\ \partial_t \bar{c}_{ij} = \frac{1}{\varepsilon} \Delta \bar{c}_{ij} + \frac{\alpha}{\varepsilon} \bar{\rho}_{ij} - \frac{\beta}{\varepsilon} \bar{c}_{ij}, \end{cases} \quad (68)$$

for $i, j = 1, 2$. The structure of system (68) is similar to (67) and therefore the estimate of the energy

$$\mathcal{E}(\bar{U}_{ij}, \bar{c}_{ij}) = \frac{1}{2} \int_{\mathbb{T}^2} [\bar{U}_{ij}^T S(\hat{U}) \bar{U}_{ij} + \lambda \bar{c}_{ij}^2] dx$$

can be performed as in step 2. The only extra terms which needs to be analyzed are the following, for $i, j, k = 1, 2$ ($C(K)$ denotes a generic constant depending on K):

$$\begin{aligned} \int_{\mathbb{T}^2} \bar{U}_{ij}^T S(\hat{U}) A_k(\hat{U})_{ij} \bar{U}_k &\leq \frac{C(K)}{\varepsilon^2} \int_{\mathbb{T}^2} |\bar{U}_{ij}| [|\underline{U}_i \underline{U}_j| + |\underline{U}_{ij}|] |\bar{U}_k| dx \\ &\leq \frac{K^2 C(K)}{\varepsilon^2} \int_{\mathbb{T}^2} |\bar{U}_{ij}| |\bar{U}_k| dx + \frac{KC(K)}{\varepsilon^2} \int_{\mathbb{T}^2} |\bar{U}_{ij}| |\underline{U}_{ij}| dx \\ &\leq \frac{(K + K^2)C(K)}{\varepsilon^2} \left[\int_{\mathbb{T}^2} |\bar{U}_{ij}|^2 dx + K^2 \right], \end{aligned}$$

where we have used once again (62) and the result in step 2. Notice that so far we have used L^∞ estimates only up to the first order derivatives of \bar{U} and \underline{U} . In the last inequality above, the second derivatives are only estimated in L^2 . We have therefore obtained, for $0 < K < 1$,

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\bar{U}_{ij}, \bar{c}_{ij}) &\leq \frac{K^3 C(K)}{\varepsilon^2} + C(K) K \frac{1}{2} \int_{\mathbb{T}^2} \left(\frac{\bar{\rho}_{ij}^2}{\varepsilon^2} + \frac{|\bar{v}_{ij}|^2}{\varepsilon^2} \right) dx \\ &\quad + \frac{\|\hat{\rho}\|_{L^\infty}}{\varepsilon^2} \int_{\mathbb{T}^2} \bar{v}_{ij} \cdot \nabla \bar{c}_{ij} dx - \frac{(\bar{\rho} - K)}{\varepsilon^2} \int_{\mathbb{T}^2} |\bar{v}_{ij}|^2 dx \\ &\quad - \frac{\lambda}{\varepsilon} \int_{\mathbb{T}^2} |\nabla \bar{c}_{ij}|^2 dx + \frac{\lambda \alpha}{\varepsilon} \int_{\mathbb{T}^2} \bar{\rho}_{ij} \bar{c}_{ij} dx - \frac{\lambda \beta}{\varepsilon} \int_{\mathbb{T}^2} \bar{c}_{ij}^2 dx \end{aligned}$$

and, by using the same choice of λ and K as in step 1, after using Gronwall Lemma we obtain

$$\mathcal{E}(\bar{U}(t), \bar{c}(t)) \leq C(K) \left[\mathcal{E}(\bar{U}(0), \bar{c}(0)) + \frac{K^3}{\varepsilon^2} \right] e^{Bt}.$$

Then, the coercivity property (66) and the assumptions (62) imply

$$\sup_{0 \leq t \leq T} \left(\|D^2 \bar{\rho}(t)\|_{L^2(\mathbb{T}^2)}^2 + \varepsilon \|D^2 \bar{v}(t)\|_{L^2(\mathbb{T}^2)}^2 + \sqrt{\varepsilon} \|D^2 \bar{c}(t)\|_{L^2(\mathbb{T}^2)}^2 \right) \leq C(K)(\delta + K^3)$$

and clearly, a choice of δ and K small enough implies $C(K)(\delta + K^3) < K^2$, which concludes the estimate of the second derivatives.

STEP 4. In order to conclude the proof of the proposition, one needs to perform the same energy estimate also on the space derivatives of order 3 and 4. All the

estimates on the nonlinear terms on the right-hand side are analogous to those in Step 3. The integrals with over-quadratic terms always contains not more than two terms involving more than two derivatives. Therefore, all the extra terms can be estimated in L^∞ by using assumption (62) and the results in the previous steps. We shall skip the details of these computations. The proof is complete. \square

We are now ready to state the main theorem of this section.

Theorem 5.1. *Let $T > 0$ and let $0 < s < 4$. Let $(\tilde{\rho}, \tilde{v}, \tilde{c})$ be the constant state in (57). There exists constants $\delta, \varepsilon_0 \in (0, 1)$ such that, if the initial data ρ_0, v_0, c_0 satisfy*

$$\|\rho_0^\varepsilon - \tilde{\rho}\|_{H^4(\mathbb{T}^2)} + \varepsilon\|v_0^\varepsilon\|_{H^4(\mathbb{T}^2)} + \sqrt{\varepsilon}\|c_0^\varepsilon - \tilde{c}\|_{H^4(\mathbb{T}^2)} \leq \delta,$$

for all $\varepsilon \in (0, \varepsilon_0)$, then there exists a classical solution $(\rho^\varepsilon, v^\varepsilon, c^\varepsilon)$ to (56) such that the quantity

$$\sup_{0 \leq t \leq T} (\|\rho^\varepsilon(t)\|_{H^s(\mathbb{T}^2)} + \varepsilon\|v^\varepsilon(t)\|_{H^s(\mathbb{T}^2)} + \sqrt{\varepsilon}\|c^\varepsilon(t)\|_{H^s(\mathbb{T}^2)})$$

is uniformly bounded with respect to $\varepsilon \in (0, \varepsilon_0)$ and such that the density ρ^ε satisfies

$$\rho^\varepsilon(x, t) > \tilde{\rho}/2 > 0$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. For any fixed $\varepsilon \in (0, \varepsilon_0)$, the sequence (ρ^n, v^n, c^n) has all space derivatives up to order 4 in L^2 and all time derivatives up to order 3 in L^2 . Therefore, (ρ^n, v^n, c^n) is relatively strongly compact in $W^{1,\infty}$ and it converge (up to a subsequence) to a solution to the original problem (56). Moreover, the estimate

$$\sup_{0 \leq t \leq T} (\|\rho^\varepsilon(t)\|_{H^s(\mathbb{T}^2)} + \varepsilon\|v^\varepsilon(t)\|_{H^s(\mathbb{T}^2)} + \sqrt{\varepsilon}\|c^\varepsilon(t)\|_{H^s(\mathbb{T}^2)}) \leq K$$

can be passed to the limit by weak lower semicontinuity and the proof is complete. \square

Remark 5. The whole procedure developed in the proof of the above theorem can be easily generalized to the case of the third scaling introduced in section 2.3.

Remark 6. We observe here that the power like expression for the pressure $g(\rho) = \rho^\gamma$ can be replaced by a more general one in order to achieve the same existence result as in the above theorem. In particular one can use $g(\rho) = \log \rho$, thus obtaining a system which relaxes toward a Keller–Segel type system with linear diffusion. Therefore, some of the relaxation results contained in chapter 4 would include Keller–Segel type system with linear diffusion as possible limits. This fact is not in contradiction with the finite time blow up phenomena occurring in the latter, because the class of initial data for which the above theorem holds is not significant enough in order to see the appearance of blow-up in the limit system. More precisely, the initial datum is not concentrated enough in order to see the appearance of a concentration in a finite time.

Acknowledgments. The authors are partially supported by the Italian research project “Sistemi iperbolici nonlineari: modelli e analisi qualitativa” of the “Istituto Nazionale di Alta Matematica - Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni” (GNAMPA 2006). MDF acknowledges support from the Wolfgang Pauli Institute of Vienna and from the Wittgenstein 2000 Award of Peter A. Markowich.

REFERENCES

- [1] D. Ambrosi, A. Gamba and G. Serini, *Cell directional and chemotaxis in vascular morphogenesis*, Bull. Math. Biol., **66** (2004), 1851–1873.
- [2] M. Burger, M. Di Francesco and Y. Dolak-Struß, *The Keller-Segel model for chemotaxis with prevention of overcrowding: Linear vs. nonlinear diffusion*, SIAM J. Math. Anal., **38** (2006), 1288–1315.
- [3] A. Blanchet, J. Dolbeault and B. Perthame, *Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations, **44** (2006), 32 pp. (electronic).
- [4] H. M. Byrne and M. R. Owen, *A new interpretation of the Keller-Segel model based on multiphase modelling*, J. Math. Biol., **49** (2004), 604–626.
- [5] V. Calvez and J. A. Carrillo, *Volume effects in the Keller-Segel model: energy estimates preventing blow-up*, J. Math. Pures Appl. (9), **86** (2006), 155–175.
- [6] J. A. Carrillo, A. Jüngel, P. A. Markowich, G. Toscani and A. Unterreiter, *Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities*, Monatsh. Math., **133** (2001), 1–82.
- [7] F. A. C. C. Chalub, P. A. Markowich, B. Perthame and C. Schmeiser, *Kinetic models for chemotaxis and their drift-diffusion limits*, Monatsh. Math., **142** (2004), 123–141.
- [8] F. A. C. C. Chalub and J. F. Rodrigues, *A class of kinetic models for chemotaxis with threshold to prevent overcrowding*, Port. Math., **63** (2006), 123–141.
- [9] S.-Y. A. Chang and P. C. Yang, *Conformal deformation of metrics on S^2* , J. Differential Geom., **27** (1988), 259–296.
- [10] B. Dacorogna, “Weak Continuity and Weak Lower Semicontinuity of Nonlinear Functionals,” Lecture Notes in Mathematics, vol. **922**, Springer-Verlag, Berlin, 1982.
- [11] M. Di Francesco and P. Marcati, *Singular convergence to nonlinear diffusion waves for solutions to the Cauchy problem for the compressible Euler equations with damping*, Math. Models Methods Appl. Sci., **12** (2002), 1317–1336.
- [12] Y. Dolak and T. Hillen, *Cattaneo models for chemosensitive movement: Numerical solution and pattern formation*, J. Math. Biol., **46** (2003), 153–170.
- [13] R. J. DiPerna, *Convergence of approximate solutions to conservation laws*, Arch. Rational Mech. Anal., **82** (1983), 27–70.
- [14] R. J. DiPerna, *Convergence of the viscosity method for isentropic gas dynamics*, Comm. Math. Phys., **91** (1983), 1–30.
- [15] R. J. DiPerna, *Compensated compactness and general systems of conservation laws*, Trans. Amer. Math. Soc., **292** (1985), 383–420.
- [16] R. J. DiPerna, *Measure-valued solutions to conservation laws*, Arch. Rational Mech. Anal., **88** (1985), 223–270.
- [17] D. Donatelli and P. Marcati, *Singular limits for nonlinear hyperbolic systems*, Evolution equations, semigroups and functional analysis (Milano, 2000), Progr. Nonlinear Differential Equations Appl., vol. **50**, Birkhäuser, Basel, 2002, 79–96.
- [18] D. Donatelli and P. Marcati, *Convergence of singular limits for multi-D semilinear hyperbolic systems to parabolic systems*, Trans. Amer. Math. Soc., **356** (2004), 2093–2121 (electronic).
- [19] J. Dolbeault and B. Perthame, *Optimal critical mass in the two-dimensional Keller-Segel model in \mathbb{R}^2* , C. R. Math. Acad. Sci. Paris, **339** (2004), 611–616.
- [20] F. Filbet, P. Laurençot and B. Perthame, *Derivation of hyperbolic models for chemosensitive movement*, J. Math. Biol., **50** (2005), 189–207.
- [21] K. O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math., **7** (1954), 345–392.
- [22] A. Gamba, D. Ambrosi, A. Coniglio, A. de Candia, S. Di Talia, E. Giraud, G. Serini, L. Preziosi and F. Bussolino, *Percolation, morphogenesis, and burgers dynamics in blood vessels formation*, Phys. Rev. Lett., **90** (2003).
- [23] H. Gajewski and K. Zacharias, *Global behaviour of a reaction-diffusion system modelling chemotaxis*, Math. Nachr., **195** (1998), 77–114.
- [24] D. Horstmann, *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences, Part I*, Jahresbericht der DMV, **105** (2003), 103–165.
- [25] D. Horstmann, *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences, Part II*, Jahresbericht der DMV, **106** (2004), 51–69.

- [26] D. Horstmann and M. Winkler, *Boundedness vs. blow-up in a chemotaxis systems*, J. Differential Equations, **215** (2005), 52–107.
- [27] M. A. Herrero and J. J. L. Velázquez, *Singularity patterns in a chemotaxis model*, Math. Ann., **306** (1996), 583–623.
- [28] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc., **329** (1992), 819–824.
- [29] R. Kowalczyk, A. Gamba and L. Preziosi, *On the stability of homogeneous solutions to some aggregation models*, Discrete Contin. Dyn. Syst. Ser. B, **4** (2004), 203–220, Mathematical models in cancer (Nashville, TN, 2002).
- [30] S. Klainerman and A. Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, Comm. Pure Appl. Math., **34** (1981), 481–524.
- [31] R. Kowalczyk, *Preventing blow-up in a chemotaxis model*, J. Math. Anal. Appl., **305** (2005), 566–588.
- [32] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol., **26** (1970), 399–415.
- [33] J. L. Lions, “Perturbations Singulières dans les Problèmes aux Limites et en Contrôle Optimal,” Lectures Notes in Math., vol. **323**, Springer-Verlag, Berlin, 1984.
- [34] C. Lattanzio and P. Marcati, *The relaxation to the drift-diffusion system for the 3-D isentropic Euler-Poisson model for semiconductors*, Discrete Contin. Dynam. Systems, **5** (1999), 449–455.
- [35] S. Luckhaus and Y. Sugiyama, *Large time behavior of solutions in super-critical cases to degenerate keller-segel systems*, Math. Model. Numer. Anal., **40** (2006), 597–621.
- [36] P. Marcati and A. Milani, *The one-dimensional Darcy’s law as the limit of a compressible Euler flow*, J. Differential Equations, **13** (1990), 129–147.
- [37] P. Marcati and R. Natalini, *Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equation*, Arch. Rational Mech. Anal., **129** (1995), 129–145.
- [38] P. Marcati and R. Natalini, *Weak solutions to a hydrodynamic model for semiconductors: The Cauchy problem*, Proc. Roy. Soc. Edinburgh Sect. A, **125** (1995), 115–131.
- [39] P. Marcati and B. Rubino, *Hyperbolic to parabolic relaxation theory for quasilinear first order systems*, J. Differential Equations, **162** (2000), 359–399.
- [40] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J., **20** (1970/71), 1077–1092.
- [41] F. Murat, *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **5** (1978), 489–507.
- [42] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl., **5** (1995), 581–601.
- [43] T. Nagai, T. Senba and K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac., **40** (1997), 411–433
- [44] C. S. Patlak, *Random walk with persistence and external bias*, Bull. of Math. Biophys., **15** (1953), 311–338.
- [45] K. Painter and T. Hillen, *Volume-filling and quorum sensing in models for chemosensitive movement*, Canadian Applied Mathematics Quaterly, **10** (2003), 280–301.
- [46] L. Tartar, *Compensated compactness and applications to partial differential equations*, Research Notes in Math., **39** (1979), 136–210.
- [47] L. Tartar, *The compensated compactness method applied to partial differential equations*, Systems of Nonlinear Partial Differential Equations (Reidel, Dordrecht) (J.M. Ball, ed.), NATO ASI, 1983.

Received January 2008; revised December 2008.

E-mail address: difrance@univaq.it

E-mail address: donatell@univaq.it