NETWORKS AND HETEROGENEOUS MEDIA
doi:10.3934/nhm.2008.3.749

Volume **3**, Number **4**, December **2008**
pp. **749–785**

LARGE TIME BEHAVIOR OF NONLOCAL AGGREGATION MODELS WITH NONLINEAR DIFFUSION

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(Communicated by Wilfrid Gangbo)

Abstract. The aim of this paper is to establish rigorous results on the large time behavior of nonlocal models for aggregation, including the possible presence of nonlinear diffusion terms modeling local repulsions. We show that, as expected from the practical motivation as well as from numerical simulations, one obtains concentrated densities (Dirac δ distributions) as stationary solutions and large time limits in the absence of diffusion. In addition, we provide a comparison for aggregation kernels with infinite respectively finite support. In the first case, there is a unique stationary solution corresponding to concentration at the center of mass, and all solutions of the evolution problem converge to the stationary solution for large time. The speed of convergence in this case is just determined by the behavior of the aggregation kernels at zero, yielding either algebraic or exponential decay or even finite time extinction. For kernels with finite support, we show that an infinite number of stationary solutions exist, and solutions of the evolution problem converge only in a measure-valued sense to the set of stationary solutions, which we characterize in detail.

Moreover, we also consider the behavior in the presence of nonlinear diffusion terms, the most interesting case being the one of small diffusion coefficients. Via the implicit function theorem we give a quite general proof of a rather natural assertion for such models, namely that there exist stationary solutions that have the form of a local peak around the center of mass. Our approach even yields the order of the size of the support in terms of the diffusion coefficients.

All these results are obtained via a reformulation of the equations considered using the Wasserstein metric for probability measures, and are carried out in the case of a single spatial dimension.

1. **Introduction.** Nonlocal models for aggregation phenomena recently received growing attention in particular in biological applications. Celebrated examples are the Keller-Segel model for chemotaxis in all its variants (cf. [23, 17, 21, 22]) and

2000 *Mathematics Subject Classification.* 35B30, 35B40, 35B41, 45K05, 92D25.
Key words and phrases. Nonlinear diffusion, nonlocal PDEs, biological aggregation, stationary solutions, asymptotic behavior, Wasserstein metric.

models for swarming of populations (cf. [4, 26, 27, 28, 31, 34]). More recently, models of the same type appeared also in the context of socio-economic applications such as opinion formation models (cf. e. g. [32]). For the problem of the justification of these models as mean field limits of large particle dynamics, we refer e.g. to [18, 28]. The mathematical analysis of such models is a challenging topic, in particular if the equation includes nonlinear diffusion terms modelling local repulsion effects (cf. [20, 28]). Some progress in this context has been made recently (cf. [3, 6, 7, 29]) using either methods of characteristics that are restricted to regular situations or entropy solution techniques that do not exploit dissipative structures and yield few information for large times. A more general framework without these shortcomings of the present approach still seems to be missing and this paper is an attempt in this direction. As has been recently pointed out in [9, 10, 1], models of this form can be formulated as metric gradient flows of certain entropy functionals on a space of probability measures equipped with the Wasserstein metric. However, detecting the long time asymptotics of the models in [9] requires a convexity assumption (the so called displacement convexity, or convexity along geodesics) on the interaction energy functional which does not apply to our case. Parallel to our work, Bertozzi and Thomas [2] proved finite time blow up in the case without diffusion with a convolution kernel having a Lipschitz point at its maximum. We also mention that Wasserstein metrics have been used also frequently in the analysis of stochastic particle models and their mean-field limits (cf. e.g. [16, 33]), indeed an exposition of the convergence of a particle model to the aggregation model considered here (without nonlinear diffusion) can be found in [18].

In this paper we restrict the analysis to the case of one spatial dimension, where the Wasserstein metric can be computed in a reasonably simple way in terms of pseudo inverses of cumulative distributions (see Section 2). Also, the gradient flow structure is much simpler in this case and the notion of convexity along geodesics turns out to be equivalent to classical convexity at the level of the pseudo–inverse equation. However, the gradient flow structure is present also in more than one space dimension. Indeed, some results could be generalized to the multidimensional case, such as the existence of more complex singular equilibria and their stability without rate up to time subsequences (cf. Theorem 3.9).

The models we consider are nonlocal parabolic evolution equations of the form

$$\partial_t \rho = \partial_x \left(\rho \partial_x [a(\rho) - G * \rho + V]\right) \qquad \text{in } \mathbb{R} \times \mathbb{R}^+, \tag{1.1}$$

with initial condition

$$\rho(.,0) = \rho_0 \geq 0 \qquad \text{in } \mathbb{R}. \tag{1.2}$$

In (1.1), $a : \mathbb{R}^+ \to \mathbb{R}$ models a nonlinear diffusion term, $G*\rho$ denotes the convolution

$$(G * \rho)(x) = \int_{\mathbb{R}} G(x-y)\rho(y)\; dy \tag{1.3}$$

with a scalar function $G : \mathbb{R} \to \mathbb{R}$, and $V : \mathbb{R} \to \mathbb{R}$ is an external potential. The nonlinear diffusion term $a(\rho)$ models local repulsions between particles, while the convolution term models wide-range attraction (and possibly also repulsion). Throughout this paper we shall require the following basic structural conditions on equation (1.1).

(SD) The function $a : [0, +\infty) \to [0, +\infty)$ belongs to $C^1((0, +\infty))$. Moreover, either a is strictly increasing such that the function $\rho \mapsto a'(\rho)$ is integrable near $\rho = 0$, or $a \equiv 0$.

(EP) $V \in C^2(\mathbb{R})$, $V', V'' \in L^\infty(\mathbb{R})$, $V(-x) = V(x)$ for all $x \in \mathbb{R}$, $V'(0) = 0$.

(IP1) $G \in W^{2,\infty}(\mathbb{R})$, $G(-x) = G(x)$ for all $x \in \mathbb{R}$, (these conditions imply automatically $G'(0) = 0$).

The assumption (IP1) will sometimes be replaced by the following weaker two assumptions.

(IP2) $G = B + C$, with B satisfying (IP1) above and $C \in C^1(\mathbb{R})$ globally concave with $C(-x) = C(x)$.

(IP3) $G \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ with $G(-x) = G(x)$; moreover, there exists a constant $\lambda \in \mathbb{R}$ such that the inequality

$$\frac{G'(x) - G'(y)}{x - y} \leq \lambda$$

holds for all $x, y \in \mathbb{R}$, $x \neq y$.

The assumption (IP2) is basically equivalent to require that the second derivative of G is bounded from above, while the assumption (IP3) allows for singularities of the second derivative of G. We shall see later on that the behavior of G'' at zero is decisive in the asymptotic behavior of solutions to (1.1) in case of absence of diffusion and without external potential V. In particular, a singular behavior of G'' at zero (in the sense of the difference quotient in (IP3) being unbounded from below near zero) may allow for finite time extinction phenomena (see Theorem 3.12).

Assumption (SD) is usually referred to as *slow diffusion*, because it implies finite speed of propagation of the support (see section 2.7). We stress here that the main difference between our models and those in [9] is that we do not require global concavity of G, which would not be coherent with biological motivations (see e.g. [6]).

Remark on the notation: In the following we shall often deal with measure valued solutions to (1.1), especially in the case $a \equiv 0$. We will use the symbol μ to denote a measure valued solution, while the symbol ρ will denote the density of μ in case the latter is absolutely continuous with respect to the Lebesgue measure.

We shall always assume that the initial measure μ_0 is scaled such that $\int_{\mathbb{R}} d\mu_0 = 1$. It is straight-forward to see (at least formally) from (1.1) that this property is conserved in time, i.e.,

$$\int_{\mathbb{R}} d\mu(t) = 1, \qquad \forall\, t \in \mathbb{R}^+. \tag{1.4}$$

Moreover, from the interpretation of ρ as a particle density it is natural to look for nonnegative solutions ρ, and hence, for all $t \in \mathbb{R}^+$, $\rho(.,t)$ can be interpreted as a probability density with associated measure μ. We are mainly interested in the study of qualitative behavior of solutions to equation (1.1) which are nonnegative measures with finite total mass and, in particular, in their asymptotic behavior for large times. The latter problem needs to be interpreted in a measure valued sense.

The paper is organized as follows: in Section 2 we recall the known existence and uniqueness theory for (1.1) (based on the gradient flow formulation of the model in the Wasserstein space of probability measures) and provide an improved existence theory for models where the assumptions on the interaction kernel G do not fit the existing theory. In Section 3 we shall prove existence of stationary solutions and their characterization in the purely aggregative case, i. e. when $a \equiv 0$. In this case we also provide existence of self–similar solutions and asymptotic stability of the stationary solutions in Wasserstein metrics. Some of the results in this section extend certain results about granular media models contained in [25] (where $-G$

is required to be convex and homogeneous). In section 4 we prove existence of compactly supported stationary solutions in case $a(\rho) = \epsilon\rho^2$ for a small enough $\epsilon > 0$.

The major results of this paper concern the structure of stationary solutions and the large-time behavior of solutions of the evolution equation (1.1). In the purely aggregative case, i.e. for $a \equiv 0$ we obtain:

- For kernels with infinite support, there is a unique stationary solution, which is a Dirac δ distribution. For kernels of finite support, there is an infinite number of stationary solutions, which are all certain linear combinations of Dirac δ distributions.
- The solutions of the evolution equation converge weakly to stationary solutions in a set-valued sense.
- For initial values sufficiently close to stationary solutions, we even provide a rate of convergence to the equilibrium depending on the behavior of G' at zero.

In the case of diffusion and aggregation, i.e. a not identically zero, we focus on the analysis of stationary solutions and provide a formal asymptotic analysis of the stationary solutions for sufficiently small diffusion. A rigorous analysis is carried out in the particularly interesting case $a(p) = \epsilon p^2$, $\epsilon > 0$, with the following results:

- For ϵ sufficiently small, there exists a stationary solution with support having a diameter of order $\epsilon^{1/3}$, i.e., one really obtains the expected peak solutions.
- For ϵ sufficiently large (compared to the L^1-norm of the kernel G), there are no stationary solutions of a similar kind.

2. **Preliminaries and existence theory.**

2.1. **Weak solutions.** Due to the presence of both nonlinear diffusion and nonlinear transport term in (1.1), one has to introduce a concept of weak solution for (1.1). Among the several possible definitions, we shall choose the one being consistent with the theory developed in [1], which consists in a gradient flow formulation in the Wasserstein space of probability measures (see Section 2.2) for the equation (1.1). We shall recall the existence and uniqueness results presented in [1] which apply to our case. Throughout the paper, $\mathcal{P}(\mathbb{R})$ will denote the space of probability measures on $\mathbb{R}$. Moreover, we denote

$$A(\rho) := \int_0^\rho \xi a'(\xi) d\xi. \tag{2.1}$$

Definition 2.1 (Presence of diffusion)**.** Suppose a in (SD) not identically zero. Then, a mapping $t \mapsto \rho(t) \in C(\mathbb{R}^+; L^1(\mathbb{R}))$ is said to be a *weak solution* to (1.1) with initial datum $\rho_0 \in L^1(\mathbb{R})$ if

$$\|G' * \rho(t) + V' + \partial_x a(\rho(t))\|_{L^2(\rho(t)dx)} \in L^2_{loc}(0, +\infty)$$

and

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}} \phi_t(x,t)\rho(x,t)dxdt - \int_{\mathbb{R}} \phi(x,0)\rho_0(x)dx \\ &= -\int_0^T \int_{\mathbb{R}} \rho a(\rho)_x \phi_x dxdt + \int_0^T \int_{\mathbb{R}} \rho G' * \rho \phi_x dxdt - \int_0^T \int_{\mathbb{R}} \rho V' \phi_x dxdt, \end{aligned} \tag{2.2}$$

for all $T > 0$ and for all $\phi \in C^\infty$ with compact support in $\mathbb{R}^+ \times [0, T)$.

In the case without diffusion we must allow for measure–valued solutions to exist, as stated in the following definition.

Definition 2.2 (Absence of diffusion)**.** Suppose $a \equiv 0$ in (SD). Then, a mapping $t \mapsto \mu(t) \in C(\mathbb{R}^+; \mathcal{P}(\mathbb{R}))$ is said to be a *weak solution* to (1.1) with initial datum $\mu_0 \in \mathcal{P}(\mathbb{R})$ if

$$\|G' * \mu(t) + V'\|_{L^2(d\mu(t)(x))} \in L^2_{loc}(0, +\infty)$$

and

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}} \phi_t(x,t) d\mu(t)(x) dt - \int_{\mathbb{R}} \phi(x,0) d\mu_0(x) \\ &= \int_0^T \int_{\mathbb{R}} G' * \mu \phi_x d\mu(t)(x) dt - \int_0^T \int_{\mathbb{R}} V' \phi_x d\mu(t)(x) dt, \end{aligned} \tag{2.3}$$

for all $T > 0$ and for all $\phi \in C^\infty$ with compact support in $\mathbb{R}^+ \times [0, T)$.

The existence of weak solutions for (1.1) in the sense of the above stated definitions can be achieved by means of the theory developed in [1]. Such theory requires the notion of p–Wasserstein distance, which we introduce in the following subsection.

2.2. **Wasserstein metric, distribution, and Pseudo-Inverse.** In the following we review the basic properties of the p–*Wasserstein distances* on the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$ (for more details see for instance the book of Villani [37] or the book of Ambrosio, Gigli and Savaré [1]). For $p > 1$ we introduce the notation

$$\mathcal{P}_p(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |x|^p d\mu(x) < +\infty \right\}.$$

For $\mu_1, \mu_2 \in \mathcal{P}_p(\mathbb{R}^d)$, the p–Wasserstein distance between μ_1 and μ_2 is defined by

$$W_p(\mu_1, \mu_2)^p = \inf \left\{ \int\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y), \ \pi \in \Pi(\mu_1, \mu_2) \right\}, \tag{2.4}$$

where $\Pi(\mu_1, \mu_2)$ is the space of all measures π on the product space $\mathbb{R}^d \times \mathbb{R}^d$ having μ_1 and μ_2 as marginal measures, i. e.

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x_i) d\pi(x_1, x_2) = \int_{\mathbb{R}^d} f(x_i) d\mu_i(x_i),$$
$$\text{for any } \mu_i\text{–integrable Borel function } f, \ i = 1, 2.$$

The set $\Pi(\mu_1, \mu_2)$ is called the set of *transference plans* between μ_1 and μ_2. The existence of an *optimal* transference plan (i.e. of a measure $\pi \in \Pi(\mu_1, \mu_2)$ such that the infimum in (2.4) is achieved) is easily proven by standard compactness properties of probability measures (see for instance [1]). If no additional assumptions on μ_1 and μ_2 are prescribed, the set $\Pi_o(\mu_1, \mu_2)$ of optimal transference plans may have more than one element. Under further regularity assumptions on one of the two measures (namely, μ_1 gives no mass to sets of finite H^{d-1} Hausdorff measure), there exists a unique optimal transference plan of the form $d\pi(x, y) = d\mu_1(x)\delta[y = T(x)]$ (i.e. π is concentrated on the graph of a map $T : \mathbb{R} \to \mathbb{R}$), where T satisfies the *push–forward* condition $T_\sharp \mu_1 = \mu_2$ which reads

$$\int \psi \circ T d\mu_1 = \int \psi d\mu_2, \qquad \text{for all} \quad \psi \in L^1(d\mu_2). \tag{2.5}$$

For future use we also recall the inequality ([37])

$$W_p(\mu,\nu) \le W_q(\mu,\nu), \qquad \text{if } \ 1 \le p \le q < +\infty. \tag{2.6}$$

2.2.1. *The one dimensional case.* In one space dimension, the optimal map T is the same for all $p > 1$ and it can be expressed in terms of the cumulative distribution functions of μ_1 and μ_2. This yields to a simplification in the expression of the p–Wasserstein distances. More precisely, let $R_i : \mathbb{R} \to [0,1]$, $i = 1,2$, be defined as the distribution function

$$R_i(x) = \mu_i((-\infty,x]), \quad i = 1,2. \tag{2.7}$$

The *pseudo-inverse* function of R_i, defined on the interval $[0,1]$, is given by

$$u_i(z) := R_i^{-1}(z) = \inf\left\{x \in \mathbb{R} \mid R_i(x) > z\right\}, \quad i = 1,2. \tag{2.8}$$

Under these notations, the p–Wasserstein metric between μ_1 and μ_2 can be expressed by

$$W_p(\mu_1,\mu_2) = \|u_1 - u_2\|_{L^p([0,1])}.$$

The previous relation can be heuristically proven as follows, at least in the case of μ_1 and μ_2 having no atoms. Since the optimal map T has to be monotone (in order to achieve the minimum rearrangement of the mass), we can take $\psi(y) = \chi_{(-\infty,x]}(y)$ in the push–forward condition (2.5) and we obtain

$$\int_{-\infty}^{T^{-1}(x)} d\mu_1(y) = \int_{-\infty}^{x} d\mu_2(y),$$

which implies $T^{-1} = R_1^{-1} \circ R_2$. Hence, we can use this expression in the definition of the Wasserstein distance

$$\begin{aligned} W_p(\mu_1,\mu_2)^p &= \int_{\mathbb{R}} |x - T(x)|^p d\mu_1(x)dx = \int_0^1 |R_1^{-1}(z) - T(R_1^{-1}(z))|^p dz \\ &= \int_0^1 |R_1^{-1}(z) - R_2^{-1}(z)|^p dz = \int_0^1 |u_1(z) - u_2(z)|^p dz. \end{aligned} \tag{2.9}$$

For the complete and rigorous proof of formula (2.9) we refer to [37, Chapter 2, Section 2]. A great advantage in using the representation formula (2.9) is that a probability measure μ can be easily represented via a function u defined on the interval $[0,1]$, which is possibly constant on certain subintervals if μ has atoms. To be more precise, consider the following example: let $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ and let F be its cumulative distribution function $F(x) = \mu((-\infty,x])$. Then, it is easy to check that the pseudo–inverse of F is given by

$$u(z) = \begin{cases} -1 & \text{if } 0 \le z < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le z \le 1. \end{cases}$$

2.2.2. *The pseudo–inverse equation.* The simplified expression (2.9) suggests writing down the explicit time evolution equation for the pseudo inverse u. Suppose for simplicity $\rho(t)$ be a solution to (1.1) such that $\rho(t)$ is smooth, positive, with connected compact support. Let $R(t)$ be its cumulative distribution function. Then $R(t) : \mathbb{R} \to [0,1]$ is invertible and its inverse $u(t) : [0,1] \to \mathbb{R}$ satisfies the following partial differential equation

$$\partial_t u(z,t) = -\partial_z\left(b(\partial_z u(z,t))\right) + \int_0^1 G'(u(z,t) - u(\zeta,t))\, d\zeta - V'(u(z,t)), \tag{2.10}$$

in $[0,1] \times \mathbb{R}^+$, where

$$b(t) := A(1/t)$$

and A is as (2.1). With such an equation at hand, one can think of estimating the p–Wasserstein distances between two solutions to our equation (1.1) in terms of direct L^p estimates of the difference between the two corresponding pseudo–inverses (i.e. L^p estimates on the equation (2.10)). However, translating the results for u into corresponding results for the original solution ρ seems to be nontrivial. To overcome this difficulty in the case a is not identically zero, one can use an approximated version of (1.1), namely

$$\partial_t \rho = \partial_x \left(\rho \partial_x [a_\lambda(\rho) - G * \rho + V]\right) \qquad \text{in } \mathbb{R} \times \mathbb{R}^+, \tag{2.11}$$

where

$$a_\lambda(\rho) = \lambda \log \rho + a(\rho), \qquad \lambda > 0$$

(the only result where such an approximation is needed is Theorem 2.12). The case $a \equiv 0$ is easier to deal with. Indeed, as we shall see later on, direct L^p estimates on the pseudo–inverse equation (2.10) do not require differentiability with respect to the z variable in this case, since no z–derivatives appear in (2.10) when $a \equiv 0$. In this case we shall provide an ad–hoc existence theory for the pseudo–inverse equation (in section 2.4) which easily allows to rephrase L^p estimates for u in terms of estimates of the p–Wasserstein distance at the level of ρ.

2.3. **Existence and uniqueness via gradient flow formulation.** In this subsection we focus on the existence and uniqueness of weak solution to (1.1). Such an issue can be easily solved via the gradient flow theory of [1]. Most of the ingredients of such a theory (which we shall recall hereafter) make sense in a multidimensional setting. However, the existence and uniqueness result below holds only in one space dimension. For future use we recall the following notions:

Definition 2.3 (Geodesics of the space $\mathcal{P}_2(\mathbb{R}^d)$)**.** A curve $[0,1] \ni t \mapsto \mu(t) \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mu(0) = \mu_1$ and $\mu(1) = \mu_2$ for given $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ is called a *constant speed geodesic* connecting μ_1 and μ_2 if

$$W_2(\mu(s), \mu(t)) = (t-s) W_2(\mu_1, \mu_2)$$

for all $0 \leq s \leq t \leq 1$.

Theorem 2.4 ([1], Theorem 7.2.2)**.** *A curve $[0,1] \ni t \mapsto \mu(t) \in \mathcal{P}_2(\mathbb{R}^d)$ is a constant speed geodesic connecting μ_1 and μ_2 if and only if it can be represented as follows:*

$$\mu(t) = R(t)_\sharp \gamma,$$

for a certain $\gamma \in \Pi_o(\mu_1, \mu_2)$ where the map $R(t) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$R(t)(x, y) = (1-t)x + ty.$$

Definition 2.5 (α–convex functionals)**.** A functional $\phi : \mathcal{P}_2 \to \mathbb{R}$ is said to be *α–convex along constant speed geodesic* for a certain $\alpha \in \mathbb{R}$ if

$$\phi(\mu(t)) \leq (1-t)\phi(\mu_1) + t\phi(\mu_2) - \frac{\alpha}{2} t(1-t) W_2^2(\mu_1, \mu_2) \tag{2.12}$$

for all $t \in [0,1]$, for any $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ and for any $\mu(t)$ constant speed geodesic connecting μ_1 and μ_2.

We can state the following existence and uniqueness theorem, which extends [1, Theorem 11.2.8] in the one dimensional case. Since we will devote all the next section to the case without diffusion, we restrict to the case a not identically zero for simplicity.

Theorem 2.6 (Existence and uniqueness [1])**.** *Suppose (SD) with a not equivalently zero, (EP) and (IP2) are satisfied. Suppose also that $a(\rho) \to +\infty$ as $\rho \to +\infty$. Then, for every $\mu_0 \in \mathcal{P}_2(\mathbb{R})$ there exists a unique distributional solution $\mu(t) = \rho(t)\mathcal{L}^1$ of (1.1) in the sense of the Definition 2.1, such that $\rho(t)\mathcal{L}^1 \to \mu_0$ narrowly as $t \to 0$ and $A(\rho(t)) \in L^1_{loc}((0,+\infty); W^{1,1}_{loc}(\mathbb{R}))$.*

Sketch of the proof. The proof of the above theorem can be carried out by means of the argument contained in the book [1], in particular in Chapter 11. Indeed, it is well known in [1] that the equation (1.1) can be interpreted as the gradient flow

$$\frac{d}{dt}\mu(t) = -\mathrm{grad}\mathcal{F}(\mu(t))$$

of the functional

$$\mathcal{F}(\mu) = \int_{\mathbb{R}} \mathcal{A}(\rho)dx + \int_{\mathbb{R}} V d\mu - \frac{1}{2}\int_{\mathbb{R}} G * \mu d\mu, \quad \mathcal{A}(\rho) := \int_0^\rho a(\xi)d\xi, \qquad \mu = \rho\mathcal{L}^1$$

in the space $\mathcal{P}_2(\mathbb{R})$ endowed with the 2–Wasserstein distance (actually, in [1] a metric space formulation of gradient flows is studied, based on an idea of De Giorgi, which turns out to be interpretable in a Riemannian sense on the space $\mathcal{P}_2$ and this justifies the use of the gradient symbol above). More precisely, the Theorem 11.2.8 of [1] ensures the existence and uniqueness of the gradient flow of $\mathcal{F}$ in $\mathcal{P}_2$ under the same assumptions (EP) and (IP) above, in any space dimension, provided that G is globally concave. In the multi dimensional case such an assumption cannot be removed because the interaction energy functional $-\frac{1}{2}\int_{\mathbb{R}} G * \mu d\mu$ seems not to satisfy a notion of generalized convexity which is needed to ensure uniqueness, unless G is concave (see the Definition 9.2.2 in [1]). On the other hand, this generalized convexity is not needed in the one dimensional case. Indeed, in this case the space $\mathcal{P}_2$ is isometrically isomorphic to a closed convex subset of an Hilbert space (namely the space of nondecreasing functions in $L^2(0,1)$) and therefore the 2–Wasserstein distance can be proven to be 2–convex along constant speed geodesics (because of the parallelogram rule). Therefore, one can apply the result in the Theorem 4.0.4 of [1] in order to prove the assertion of Theorem 2.6 provided the functional $\mathcal{F}$ above is α–convex for some real number α. This is proven in the following Lemma.

Lemma 2.7. *Under the assumptions (SD), (EP) and (IP2), the functional $\mathcal{F}$ defined before is α–convex with $\alpha = -\|V''\|_{L^\infty} - 2\|B''\|_{L^\infty}$.*

Proof. It is already well known that the free energy part $\int_{\mathbb{R}} \mathcal{A}(\rho)dx$ is 0–convex along constant speed geodesics and that the potential energy $\int_{\mathbb{R}} V(x)d\mu$ is α–convex with $\alpha = -\|V''\|_{L^\infty}$. An analogous property holds for the interaction energy $-\frac{1}{2}\int_{\mathbb{R}} G * \mu d\mu$. A similar property was proven in [10]. We reproduce here the proof for the sake of completeness.

Consider then a constant speed geodesic $\mu(t)$ connecting μ_1 and μ_2, represented by the formula

$$\mu(t) = R(t)_\sharp \gamma,$$

for a certain $\gamma \in \Pi_o(\mu_1, \mu_2)$ and with $R(t)(x,y) = (1-t)x + ty$. Due to the assumption (IP'), the map

$$\mathbb{R} \ni z \mapsto -G(z) + \frac{\beta}{2} z^2, \qquad \beta = \|B''\|_{L^\infty}$$

is convex. Therefore, by the parallelogram rule, it is easy to see that

$$-G((1-t)x + ty) \le (1-t)G(x) + tG(y) + \frac{\beta}{2} t(1-t)(x-y)^2,$$

and this implies

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G(x-y)d\mu(t)(y)d\mu(t)(x)\\
&= -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} G((1-t)x_1 + tx_2 - (1-t)y_1 - ty_2)d\gamma(x_1,x_2)d\gamma(y_1,y_2)\\
&\le \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \Big[-(1-t)G(x_1-y_1) - tG(x_2-y_2)\\
&\quad + \frac{\beta}{2} t(1-t)[(x_1-y_1) - (x_2-y_2)]^2 \Big] d\gamma(x_1,x_2)d\gamma(y_1,y_2)\\
&= -\frac{(1-t)}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G(x-y)d\mu_1(x)d\mu_1(y) - \frac{t}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G(x-y)d\mu_2(x)d\mu_2(y)\\
&\quad + \beta t(1-t)W_2^2(\mu_1,\mu_2).
\end{aligned}$$

□

Hence, the proof of Theorem 2.6 is complete.

Remark 2.8. The above result is still valid under a much weaker assumption on G and V, namely $-G$ and V being α–convex.

For future use we also recall the following energy identity

$$\mathcal{F}(\rho(t)) + \int_0^t \|G' * \rho(s) + V' + \partial_x a(\rho(s))\|^2_{L^2(\rho(s)dx)} ds = \mathcal{F}(\rho(0)), \tag{2.13}$$

for almost every $t > 0$, which can be obtained as byproduct of the theory developed in ([1]).

2.4. **Improved existence theory in absence of diffusion.** The aim of this section is to rephrase (and, in a certain sense, improve) the existence theory in the case $a \equiv 0$ in (1.1). As pointed out in the subsection 2.2.2, in this case we are interested in the pseudo–inverse equation

$$\partial_t u(z,t) = \int_0^1 G'(u(z,t) - u(\zeta,t))d\zeta - V'(u(z,t)), \tag{2.14}$$

for almost every $z \in (0,1)$ and for all $t \ge 0$. Our idea is to prove that the equation (2.14) has solutions which are differentiable almost everywhere in t, in such a way that one can perform direct L^p estimates on it. Since all the results on the qualitative behavior of (2.14) will concern with compactly supported initial data, we shall work under the assumption $u_0 \in L^\infty([0,1])$ on the initial datum u_0. As a byproduct of that, we shall obtain the finite rate of propagation of the support as a direct consequence of our existence theory (see the Section 2.7).

In the following theorem we deal with the case of G satisfying the assumption (IP1).

Theorem 2.9. *Suppose that G and V satisfy (IP1) and (EP) respectively. Let $u_0 \in L^\infty([0,1])$ be nondecreasing. Then, there exists a unique solution*

$$u(t) \in \mathrm{Lip}_{loc}([0,+\infty), L^\infty([0,1]))$$

to (2.14) such that $u(0) = u_0$ and $z \mapsto u(z,t)$ is nondecreasing for all $t \geq 0$. Moreover,

(i) *the function*

$$F(x,t) := \inf\{z \in [0,1] \ : \ u(z,t) > x\}, \qquad x \in \mathbb{R}, \quad t > 0$$

has bounded variation with respect to x for almost all $t > 0$;

(ii) *the measure valued derivative $\mu(t) := \partial_x F(\cdot,t)$ is a compactly supported weak solution to the equation (1.1) with $a \equiv 0$ (in the sense of definition 2.2) with initial datum $\mu_0 := \partial_x F(\cdot,0)$;*

(iii) *if the initial measure μ_0 has a compact and connected support, is absolutely continuous with respect to $\mathcal{L}^1$ with density ρ_0 continuous and bounded such that $\rho_0(x) > 0$ as x is an interior point of $\mathrm{supp}(\mu_0)$, then so is the solution $\mu(t)$ at any time $t > 0$. In particular, no finite time concentration phenomena are possible.*

Proof. Step 1 (local existence). This step follows the proof of the Cauchy–Lipschitz existence theorem on the space $L^\infty(0,1)$. For a fixed time $T > 0$ consider the Banach space $B_T := L^\infty([0,1] \times [0,T])$ endowed with the usual L^∞ norm on the rectangle $[0,1] \times [0,T]$. The operator

$$(\mathcal{T}u)(z,t) := u_0(z) + \int_0^t \int_0^1 G'(u(z,s) - u(\zeta,s))d\zeta ds - \int_0^t V'(u(z,s))ds \tag{2.15}$$

is clearly well defined as a map from B_T into itself. Moreover, for given $u, v \in B_T$ one can easily prove the estimate

$$|(\mathcal{T}u)(z,t) - (\mathcal{T}v)(z,t)| \leq T(2\|G''\|_{L^\infty} + \|V''\|_{L^\infty})\|u - v\|_{L^\infty([0,1]\times[0,T])},$$

which implies that $\mathcal{T}$ is a contraction on B_{T^*} for small $T^* > 0$. Therefore, Banach's fixed point Theorem ensures the existence of $u \in B_{T^*}$ such that $(\mathcal{T}u)(z,t) = u(z,t)$ almost everywhere in $(z,t) \in [0,1] \times [0,T^*]$. Moreover, the assumptions on G and V ensure that u is a Lipschitz function with respect to t on the time interval $[0,T]$ and it satisfies (2.14) for almost al $(z,t) \in [0,1] \times [0,T^*]$.

Step 2 (global existence). It is clear that the boundedness of the L^∞ norm of u on any rectangle $[0,1] \times [0,T]$ for an arbitrary $T > 0$ implies global existence for $u \in L^\infty([0,T); L^\infty([0,1]))$ for all $T > 0$. To prove such a global control, for $p > 1$ we perform the estimate

$$\begin{aligned}
\frac{d}{dt}\int_0^1 |u(z,t)|^p dz &= p\int_0^1 \mathrm{sign}(u(z,t))|u(z,t)|^{p-1} u_t(z,t)dz \\
&= p\int_0^1\int_0^1 \mathrm{sign}(u(z,t))|u(z,t)|^{p-1} G'(u(z,t) - u(\zeta,t))d\zeta dz \\
&\quad - p\int_0^1 \mathrm{sign}(u(z,t))|u(z,t)|^{p-1} V'(u(z,t))dz \\
&\leq p(2\|G''\|_{L^\infty} + \|V''\|_{L^\infty})\int_0^1 |u(z,t)|^p dz,
\end{aligned}$$

where we have used Hölder inequality and the assumptions on G and V. The above estimate implies

$$\|u(t)\|_{L^p([0,1])} \leq e^{(2\|G''\|_{L^\infty}+\|V''\|_{L^\infty})t}\|u_0\|_{L^p([0,1])},$$

for all $p \in [1,+\infty)$ and all $t \in [0,T^*]$. By sending $p \to +\infty$ we obtain

$$\|u\|_{L^\infty([0,1]\times[0,T])} \leq e^{(2\|G''\|_{L^\infty}+\|V''\|_{L^\infty})T}\|u_0\|_{L^\infty([0,1])}$$

which proves the desired global bound.

STEP 3 (FURTHER REGULARITY). From the identity

$$\begin{aligned}
&\frac{d}{dt}\left[u(z_1,t) - u(z_2,t)\right] \\
&= u_0(z_1) - u_0(z_2) + \int_0^t \int_0^1 [G'(u(z_1,\tau) - u(\xi,\tau)) - G'(u(z_2,\tau) - u(\xi,\tau))]d\xi d\tau \\
&- \int_0^t [V'(u(z_1,\tau)) - V'(u(z_2,\tau))]d\tau,
\end{aligned}$$

one easily recovers the following estimates for the Lipschitz semi-norm of u

$$\frac{d}{dt}[u(t)]_{Lip} \leq [u_0]_{Lip} + [\|G''\|_{L^\infty} + \|V''\|_{L^\infty}]\int_0^t [u(\tau)]_{Lip}d\tau,$$

which implies $[u(t)]_{Lip} \leq [u_0]_{Lip}e^{[\|G''\|_{L^\infty}+\|V''\|_{L^\infty}]t}$. Therefore, if u_0 is Lipschitz, so is $u(t)$ at any $t > 0$.

STEP 4 (APPROXIMATION BY CONTINUOUS SOLUTIONS). Let $u_{0,n}$ be a sequence in $C_b([0,1])$ such that $u_{0,n}$ converges to $u_0 \in L^\infty([0,1])$ in L^p for some $p > 1$. Let u_n be the solution with initial datum $u_{0,n}$ and u be the solution with initial datum u_0. By estimating directly the time derivative of $\|u_n(t) - u(t)\|^p_{L^p([0,1])}$ as in Step 2, one can easily prove the inequality

$$\|u_n(t) - u(t)\|_{L^p([0,1])} \leq \|u_{0,n} - u_0\|_{L^p([0,1])}e^{(2\|G''\|_{L^\infty}+\|V''\|_{L^\infty})t},$$

which implies in particular that $u_n(t)$ converges to $u(t)$ in L^p.

STEP 5 (MONOTONICITY). We prove next that if the initial datum u_0 is nondecreasing (as in the assumptions), then so is the solution $u(t)$ for any $t > 0$. Let us first consider the case of an initial datum u_0 with Lipschitz regularity in z. Then, we know that the solution $u(t)$ is also Lipschitz in z (from Step 3). Let $z \in [0,1]$ and let $h \in \mathbb{R}\setminus\{0\}$ small enough such that $z+h \in [0,1]$. Let $w(h,t) := u(z+h,t) - u(z,t)$. Then, we know that $w(h,0)/h \geq 0$ for all $h \neq 0$. We compute (still for $h \neq 0$)

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{w(h,t)}{h} &= \frac{1}{h}\int_0^1 \left[G'(u(z+h,t) - u(\zeta,t)) - G'(u(z,t) - u(\zeta,t))\right] d\zeta \\
&- \frac{1}{h}\left[V'(u(z+h,t)) - V'(u(z,t))\right] \\
&\geq -\frac{1}{h}\left[\|G''\|_{L^\infty} + \|V''\|_{L^\infty}\right] w(h,t),
\end{aligned}$$

which implies

$$\frac{\partial}{\partial t}\left\{\frac{w(h,t)}{h}e^{[\|G''\|_{L^\infty}+\|V''\|_{L^\infty}]t}\right\} \geq 0 \tag{2.16}$$

and this proves that $w(h,t)/h \geq 0$ for all $t > 0$ and therefore $u(t)$ is nondecreasing. To deal with the case of a general nondecreasing and bounded initial datum u_0, we take a sequence of Lipschitz continuous initial data $u_{0,n}$ converging to u_0 in some L^p, $p > 1$, and we perform the above computation for the approximating solution

$u_n(t)$. By Step 4, the sequence $u_n(t)$ (of nondecreasing functions) converges in L^p to the solution $u(t)$ with initial datum u_0. Hence, $u(t)$ is also nondecreasing (on a set of measure 1). Moreover, it is clear from (2.16) that $u(t)$ remains strictly monotone for all times t if initially so.

STEP 6 (PASSING TO (1.1)). Since $u(\cdot,t)$ is nondecreasing, so is $F(\cdot,t)$ and therefore $F(\cdot,t)$ has bounded variation. Hence, the distributional derivative $\mu(t) = \partial_x F(\cdot,t)$ is well defined as a probability measure on $\mathbb{R}$. Now, take a sequence of C^1 and strictly increasing initial data $u_{0,n}$ which converge almost everywhere and in $L^2([0,1])$ to our initial datum u_0. We already know from the argument in Step 4 that the corresponding sequence of solutions u_n converges in $L^2([0,1]\times[0,T])$ to u for all $T>0$. Due to the assumptions on G and V and by a direct use of the equation (2.14) one easily deduces that $\partial_t u_n$ converges to $\partial_t u$ in $L^2([0,1]\times[0,T])$. By extracting a subsequence (still denoted by u_n for simplicity) we obtain for any $T>0$

$$u_n \to u \quad \text{in} \quad H^1([0,T],L^2([0,1])) \quad \text{and} \quad \text{a. e. in} \quad [0,1]\times[0,T].$$

Let us consider now a test function $\phi\in C_c^\infty(\mathbb{R}\times[0,T))$. By multiplying the equation (2.14) for $u_n(z,s)$ by $\phi_x(u_n(z,s),s)$ and by integrating over $(z,s)\in[0,1]\times[0,t]$ we obtain

$$\begin{aligned}
&\int_0^t\int_0^1 \phi_x(u_n(z,s),s)\partial_s u_n(z,s)dzds\\
&=\int_0^t\int_0^1\int_0^1 G'(u_n(z,s)-u_n(\zeta,s))\phi_x(u_n(z,s),s)d\zeta dzds\\
&-\int_0^t\int_0^1 V'(u_n(z,s))\phi_x(u_n(z,s),s)dzds. \qquad (2.17)
\end{aligned}$$

From the arguments in Steps 3 and 5 it is clear that $u_n(\cdot,t)$ is C^1 and strictly increasing at any time $t>0$. Therefore, the corresponding inverse $F_n(\cdot,t)$ is also C^1 and strictly increasing on a bounded interval. Hence, the space derivative $\partial_x F_n$ is absolutely continuous with respect to $\mathcal{L}^1$, with a density ρ_n. Now, since u_n is continuous and strictly increasing the pseudo inverse F_n of u_n satisfies $u_n(F_n(x,t),t)=x$ for all $x\in u_n^{-1}((0,1))$. By differentiating such an identity with respect to t (which is possible almost everywhere), we obtain

$$\partial_t u_n(F_n(x,t),t) = -\partial_t F_n(x,t)(\partial_x F_n(x,t))^{-1}.$$

We can then change variable $z=F_n(x,t)$ in (2.17) and obtain

$$\begin{aligned}
&-\int_0^t\int_{\mathbb{R}} \phi_x(x,s)\partial_s F_n(x,s)dxds = \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} G'(x-y)\phi_x(x,s)\rho_n(y,s)\rho_n(x,s)dydxds\\
&-\int_0^t\int_{\mathbb{R}} V'(x)\phi_x(x,s)\rho_n(x,s)dxds.
\end{aligned}$$

By Step 4 and by standard properties of the Wasserstein distance (see e.g. [1, Proposition 7.1.5]), we deduce that $\rho_n\mathcal{L}^1$ is tight, and therefore the above identity can be sent to the limit as $n\to+\infty$ (after integration by parts in the left hand side) to obtain the weak formulation (2.3) for the limit measure $\mu(t)$.

STEP 7 (ABSENCE OF CONCENTRATION). If μ_0 is as in the assumption in (iii), then its cumulative distribution F_0 is C^1 and strictly increasing on the support of μ_0 and therefore its pseudo–inverse u_0 is C^1, strictly increasing and with a strictly positive z–derivative. Due to Steps 3 and 5 we then deduce that $u(t)$ satisfies the same properties, and this easily implies the statement of (iii). $\square$

Remark 2.10. The non concentration property in point (iii) is a clear consequence of the C^2 regularity of the kernel G. In the next theorem we generalize the existence and uniqueness results in case G may have singularities in the second derivative. Later on, in Theorem 3.12, we shall see that the property (iii) is in general not satisfied under the assumptions of the following theorem, whence finite time concentration may happen.

Theorem 2.11 (Existence and uniqueness for singular interaction kernels)**.** *Suppose that G and V satisfy (IP3) and (EP) respectively. Let $u_0 \in L^\infty([0,1])$ be nondecreasing. Then, all the statements in Theorem 2.9 are true except for (iii).*

Proof. We prove this theorem by assuming $V \equiv 0$ for simplicity, because the presence of the external potential in this theorem does not bring any particular difficulty to the existence and uniqueness problem. We consider a family of interaction kernels $G_\epsilon \in W^{2,\infty}(\mathbb{R})$, with $\epsilon > 0$, such that

- $G_\epsilon'(z) \to G'(z)$ as $\epsilon \searrow 0$ for all $z \in \mathbb{R}$,
- $\{G_\epsilon'\}_\epsilon$ is uniformly bounded in L^∞,
- $G_\epsilon'' \leq 2\lambda$ for all $\epsilon > 0$ (λ is given by the assumption (IP3)).

For every fixed $\epsilon > 0$, the equation

$$\partial_t u^\epsilon(z,t) = \int_0^1 G_\epsilon'(u^\epsilon(z,t) - u^\epsilon(\zeta,t))d\zeta, \qquad z \in [0,1],\ t \geq 0, \tag{2.18}$$

admits a unique solution u^ϵ satisfying the statements of the Theorem 2.9. In particular, for a fixed time $T > 0$, by means of a direct L^2 estimate on (2.18), one can easily prove that $\{u^\epsilon\}_{\epsilon>0}$ is uniformly bounded in $L^2([0,1]\times[0,T])$. Therefore, (2.18) directly says that $\partial_t u^\epsilon$ is also uniformly bounded in the same space. Now, for $h > 0$, $h \ll 1$ let us consider a smooth cut-off function $\chi_h(z)$ such that $\chi_h(z) = 1$ for $z \in [2h, 1-2h]$, $\chi_h(z) = 0$ for $z \in [0,h] \cup [1-,1]$ and $\chi_h(z) \in [0,1]$ for all $z \in [0,1]$. Let us fix an $h \in \mathbb{R}$ and compute

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \chi_h(z)[u^\epsilon(z+h,t) - u^\epsilon(z,t)]^2 dz \\
&= 2\int_0^1\int_0^1 \chi_h(z)[u^\epsilon(z+h,t) - u^\epsilon(z,t)][G_\epsilon'(u^\epsilon(z+h,t) - u^\epsilon(\zeta,t)) - \\
&G_\epsilon'(u^\epsilon(z,t) - u^\epsilon(\zeta,t))]d\zeta dz \leq 4\lambda \int_0^1 \chi_h(z)[u^\epsilon(z+h,t) - u^\epsilon(z,t)]^2 dz,
\end{aligned} \tag{2.19}$$

where we have used the mean value theorem for G_ϵ'. The previous estimate implies, after sending $h \to 0$,

$$\|u^\epsilon(\cdot+h,t) - u^\epsilon(\cdot,t)\|_{L^2([0,1])} \leq e^{4\lambda t}\|u_0(\cdot+h) - u_0(\cdot)\|_{L^2([0,1])}$$

and the right hand side above tends to zero as $h \to 0$ uniformly with respect to ϵ. Then, the well known Riesz–Frechet–Kolmogorov L^p–compactness criterion implies that $\{u^\epsilon\}_\epsilon$ is strongly compact in $L^2([0,1],[0,T])$, and it therefore admits a subsequence converging almost everywhere to a $u \in L^2([0,1],[0,T])$. Then, it is easy to conclude that u satisfies (2.14) and that $u \in L^\infty([0,1],[0,T])$ if u_0 is bounded. The monotonicity of u is a consequence of the monotonicity of u^ϵ and of the convergence almost everywhere of u^ϵ. Uniqueness can be easily proven by computing

$$\frac{d}{dt}\int_0^1 [u_1(z,t) - u_2(z,t)]^2 dz$$

and proceeding as in (2.19) for two given solutions u_1 and u_2 having the same initial datum. All the other properties of u can be proven as for the Theorem 2.9. □

2.5. **An example of non–uniqueness.** Let us take $a \equiv 0$ and $V \equiv 0$ in (1.1), and let us consider the interaction kernel $G(z) = |z|^{\alpha+1}$ for a certain $\alpha \in (0,1)$. Clearly, this kernel violates all conditions (IP1), (IP2), (IP3), since its second derivative blows up to $+\infty$ at zero. Consider the initial datum

$$\mu_0 = \delta_0.$$

We shall prove here that the Cauchy problem

$$\begin{cases} \partial_t \mu(t) = -\partial_x(\mu \partial_x G * \mu) \\ \mu(0) = \mu_0 \end{cases}$$

has more than one solution in the sense of definition 2.2. In order to see that, let us consider the corresponding problem for the pseudo–inverse variable $u : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$,

$$\begin{cases} u_t = \displaystyle\int_0^1 G'(u(z,t) - u(\zeta,t)) d\zeta \\ u(0) \equiv 0. \end{cases}$$

This problem has the trivial solution $u(t) \equiv 0$ for all $t \geq 0$, which corresponds to the solution $\mu(t) = \delta_0$ for all $t \geq 0$. We shall prove that it is also possible to construct a solution of the form

$$u(z,t) = \begin{cases} -m(t) & \text{if } \ z \in [0, 1/2) \\ m(t) & \text{if } \ z \in [1/2, 1], \end{cases} \tag{2.20}$$

for a certain function $t \mapsto m(t) > 0$, and this corresponds to the measure valued solution

$$\mu = \frac{1}{2}\left(\delta_{-m(t)} + \delta_{m(t)}\right)$$

to (1.1) in the sense of definition 2.2. We plug (2.20) in the above equation for u at a point $z \in (1/2, 1]$ and we get

$$\dot{m}(t) = \frac{1}{2} G'(2m(t)) = (\alpha+1) 2^{\alpha-1} m(t)^\alpha, \tag{2.21}$$

and the same equation can be obtained by considering $z \in [0, 1/2)$. It is well known that the ordinary differential equation (2.21) with initial datum $m(0) = 0$ admits a nontrivial solution $t \mapsto m(t)$ (indeed, it has infinite nontrivial solutions). This concludes the proof.

We remark that the same technique will be used later on in order to prove existence of self–similar measure valued solutions. We also remark that the interaction kernel considered here is *repulsive*, i. e. globally convex. Therefore, the interpretation of this non–uniqueness phenomenon is very simple: the stationary solution given by the delta measure at zero is so unstable that it can split in two atoms at any time, and this is due to the very strong repulsive behavior of the kernel at zero. This situation is opposite to many cases in this paper where the kernel G will be concave, at least in a neighborhood of its unique stationary point. Indeed, we shall see in Theorem 3.12 that if $G(z)$ behaves like $-|z|^{\alpha+1}$ near zero (i. e. the same behavior as above but with the opposite sign), the solution concentrates to δ_0 in finite time.

2.6. **Conserved quantities.** Let us now take a closer look on conserved quantities of (1.1) and (2.10), respectively. As we have noticed above, the total mass $\int_{\mathbb{R}} \rho\, dx$ is conserved during the evolution, which was the basis of introducing the distribution function and its pseudo-inverse. Consequently there is no equivalent conservation property of the model (2.10), it is somehow hidden in the fact that one can always consider u as a function on the interval $[0, 1]$. In absence of an external potential V, the solutions also conserve the center of mass, given by the first moment of the density (if it exists)

$$CM := \int_{\mathbb{R}} \rho x\ dx = \int_{\mathbb{R}} x\ d\mu_\rho(x), \tag{2.22}$$

which can be seen as follows: taking the test function x for (1.1) we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} \rho(x,t)x\ dx &= \int_{\mathbb{R}} \partial_t\rho(x,t)x\ dx \\ &= -\int_{\mathbb{R}} \rho\partial_x[a(\rho) - G * \rho]\ dx \end{aligned}$$

Now $\rho\partial_x a(\rho) = \partial_x \tilde{a}(\rho)$ for a function $\tilde{a}$ with derivative $\tilde{a}'(p) = pa'(p)$, and thus, the first term integrates to zero. The second term is

$$\int_{\mathbb{R}} \rho G' * \rho\ dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \rho(x)G'(x-y)\rho(y)\ dy\ dx$$

and from the symmetry of G we deduce

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} \rho(x)G'(x-y)\rho(y)\ dy\ dx \\ &= -\int_{\mathbb{R}}\int_{\mathbb{R}} \rho(y)G'(y-x)\rho(x)\ dy\ dx = -\int_{\mathbb{R}} \rho(G' * \rho)\ dy, \end{aligned}$$

and hence, the second term vanishes, too. In terms of the pseudo-inverse, the center of mass can be rewritten as

$$CM = \int_0^1 u\ dz, \tag{2.23}$$

which means that the mean value of u is conserved in time if there is no external potential.

2.7. **Finite speed of propagation.** The structural condition (SD) (with a not identically zero) on the nonlinear diffusion term in (1.1) implies a *slow* propagation of the support of the solution, in the same fashion as in the porous medium equation (see [35, 36]). Following the ideas in [8, 7], one can prove that the supports of the solutions propagate with finite speed by using an interpretation of the limit as $p \to +\infty$ of the p–Wasserstein distances. Let us first introduce the ∞–Wasserstein distance (see also [11])

$$W_\infty(\mu_1, \mu_2) := \lim_{p\to+\infty} W_p(\mu_1, \mu_2),$$

the definition of which is justified by (2.6). From (2.9) it is clear that

$$W_\infty(\mu_1, \mu_2) = \|u_1 - u_2\|_{L^\infty([0,1])}.$$

Moreover, the following estimate on the speed of propagation of the support of two compactly supported measures μ_1, μ_2 can be easily proven:

$$\max\{|\inf(\text{supp}\mu_1) - \inf(\text{supp}\mu_2)|, |\sup(\text{supp}\mu_1) - \sup(\text{supp}\mu_2)|\} \\ \leq W_\infty(\mu_1, \mu_2). \tag{2.24}$$

We can state the following result, the proof of which follows the proof of a similar theorem in [7], which is in its turn inspired by the paper [8]. Therefore, this proof will be only sketched. We remark once again that the finite rate of propagation property has been already proven in case of absence of diffusion (see e.g. Theorem 2.9).

Theorem 2.12. *Suppose G satisfies (IP1) and V satisfies (EP). Let ρ be a solution to (1.1) having a compactly supported initial datum ρ_0. Then, the support of $\rho(t)$ at any time $t > 0$ is compact.*

Sketch of the proof. We consider the non degenerate approximation (2.11) for $\lambda > 0$ on the domain $(x,t) \in [-R,R] \times [0,+\infty)$ with zero flux boundary conditions

$$\partial_x[a_\lambda(\rho) - G * \rho + V] = 0 \qquad x = \pm R.$$

Due to its nondegenerate parabolic nature, the equation (2.11) enjoys at least as much space regularity as its initial datum ρ_0 (see e. g. [24] and a similar regularity result obtained via Schauder Fixed point in [7]). Moreover, the result in Lemma A.1 in the Appendix ensures that, if $\rho \geq \mu > 0$, then the solution $\rho(t)$ remains strictly positive for all $t > 0$. Therefore, the passage from the pseudo inverse equation (2.10) (with b replaced by the corresponding λ depending function) to the equation (2.11) does not present any difficulty in this case, since the primitive of $\rho(t)$ is strictly increasing and its inverse $u(t)$ enjoys enough regularity in order to perform direct L^p estimates in the usual way (see Step 2 in the proof of the Theorem 2.9). Hence, we can proceed as in [7, Theorem 3.10] and control

$$\|u(t) - \bar{u}_\lambda(t)\|_{L^p}, \qquad p \in (2, +\infty),$$

where $\bar{u}$ is the solution to the pseudoinverse equation related to the nonlinear diffusion equation

$$\rho_t = (\rho(a_\lambda(\rho)_x)_x$$

with the same initial datum ρ_0. We can then send $\lambda \to 0$ and use standard weak L^p compactness of the family of solutions u_λ to (2.11) (see [36]) and use the lower semicontinuity of the p–Wasserstein distances with respect to weak L^1 convergence to obtain

$$\|\bar{u}(t) - u(t)\|_p < \infty,$$

where $\bar{u} = \lim_{\lambda \to +\infty} u_\lambda$. Then, by sending $p \to +\infty$ and by using the fact that $\bar{u}$ is the solution to a nonlinear degenerate diffusion equation with compactly supported initial data, we easily conclude the proof by recalling that $\|\bar{u}\|_\infty < \infty$, as it is well known by classical results on the support of solutions to nonlinear diffusion equations [36].

2.8. **Alternative (formal) gradient flow formulation.** In this subsection we remark that the gradient flow formulation in terms of the Wasserstein metric turns out to be a simple gradient flow for the pseudo-inverse in a classical Hilbertian framework. In particular, in this case one does not need to introduce the notion of convexity along geodesics, which will be proven to be (formally) equivalent to classical convexity at the level of the pseudo–inverse variable u. However, most of the contents of this section are only formal, in the sense that we shall not reproduce an alternative existence and uniqueness proof for (1.1).

We start by observing that the equation for the pseudo-inverse (2.10) can be written as a standard gradient flow of the form

$$\partial_t u = -E'[u], \tag{2.25}$$

where the functional E is defined as follows. Let $B : \mathbb{R} \to \mathbb{R}$ be a primitive of b, i.e., $B' = b$. Then we define

$$E[u] := \int_0^1 \left[B\left(\partial_z u(z)\right) + V(u(z))\right]\, dz - \frac{1}{2}\int_0^1\int_0^1 G(u(z)-u(\zeta))\, d\zeta\, dz. \tag{2.26}$$

Note that E is well-defined at least for $u \in C^1([0,1])$ such that $B(\partial_z u) < \infty$ in $[0,1]$. On this set, the directional derivative of E is well defined and given by

$$\begin{aligned} E'[u]v \quad = \quad & \int_0^1 \left[b\left(\partial_z u(z)\right)\partial_z v(z) + V'(u(z))v(z)\right]\, dz + \\ & -\frac{1}{2}\int_0^1\int_0^1 G'(u(z)-u(\zeta))(v(z)-v(\zeta))\, d\zeta\, dz. \end{aligned} \tag{2.27}$$

Exploiting the symmetry of G (and the consequent anti-symmetry of G') we obtain

$$-\frac{1}{2}\int_0^1\int_{\mathbb{R}} G'(u(z)-u(\zeta))(v(z)-v(\zeta))\, d\zeta\, dz = -\int_0^1\int_0^1 G'(u(z)-u(\zeta))\, d\zeta\; v(z)\, dz.$$

After integration by parts, we can identify the derivative of E with

$$E'[u] = -\partial_z\left(b(\partial_z u(z))\right) + V'(u(z)) - \int_0^1 G'(u(z)-u(\zeta))\, d\zeta,$$

i.e., (2.10) can indeed be formulated as a gradient flow (2.25).

As one can easily expect (due to what has been proven in section 2.3), the functional E defined above is α–convex in the classical sense, i. e.

$$E[tu+(1-t)v] \leq tE[u]+(1-t)E[v]-\frac{\alpha}{2}t(1-t)\|u-v\|_2^2 \qquad \forall\, u,v\in L^2([0,1]), \tag{2.28}$$

for a suitable constant α and for all $t \in (0,1)$. In order to prove such assertion, we split the energy functional in the form $E = E_1 + E_2$ with

$$E_1[u] := \int_{\mathbb{R}} B\left(\partial_z u(z)\right)\, dz \tag{2.29}$$

and

$$E_2[u] := \int_{\mathbb{R}} V(u(z))\, dz - \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G(u(z)-u(\zeta))\, d\zeta\, dz. \tag{2.30}$$

Due to the convexity of B we immediately obtain that E_1 is a convex functional, and therefore we concentrate our attention on the properties of E_2.

Lemma 2.13. *Let G and V be twice continuously differentiable with bounded derivatives. Then the functional E_2 is twice continuously Frechet-differentiable on $L^2([0,1])$ and there exists a constant C such that*

$$E_2''[u](\varphi,\varphi) \geq -C\|\varphi\|^2 \tag{2.31}$$

Proof. We can compute

$$E_2'[u]\varphi = \int_{\mathbb{R}} V'(u(z))\varphi(z)\, dz - \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G'(u(z)-u(\zeta))(\varphi(z)-\varphi(\zeta))\, d\zeta\, dz$$

and

$$
\begin{aligned}
E_2''[u](\varphi,\psi) &= \int_{\mathbb{R}} V''(u(z))\varphi(z)\psi(z)\;dz - \\
&\quad \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G''(u(z)-u(\zeta))(\varphi(z)-\varphi(\zeta))(\psi(z)-\psi(\zeta))\;d\zeta\;dz.
\end{aligned}
$$

By standard estimates one can verify that E_2' and E_2'' are Frechet-derivatives, and one obtains (2.31) with $C = \|V''\|_\infty + 2\|G''\|_\infty$. $\square$

As a consequence of (2.31) we have (with the notation $w = tu + (1-t)v$)

$$
\begin{aligned}
E_2[v] &= E_2[w] + tE_2'[w](v-u) + \frac{t^2}{2}\int_0^1 E_2''[v+\sigma t(v-u)](v-u,v-u)\;d\sigma \\
&\geq E_2[w] + tE_2'[w](v-u) - \frac{Ct^2}{2}\|v-u\|_2^2,
\end{aligned}
$$

and by analogous reasoning we obtain

$$
E_2[u] \geq E_2[w] + (1-t)tE_2'[w](u-v) - \frac{C(1-t)^2}{2}\|v-u\|_2^2.
$$

Taking a convex combination of the last two inequalities we obtain

$$
E_2[w] \leq tE_2[u] + (1-t)E_2[v] + C(1-t)^2\|v-u\|_2^2.
$$

Hence, adding the convex functional E_1, which satisfies

$$
E_1[w] \leq tE_1[u] + (1-t)E_1[v],
$$

we obtain (2.28) with $\alpha = -2C$.

We are confident that from the α-convexity (2.28) one can deduce existence and uniqueness of solutions of the gradient flow (2.25) by specialization of general results in the classical theory of Hilbertian gradient flows. Since we already have a satisfactory existence and uniqueness theory for our models, we shall skip the rigorous analysis of this problem.

3. Large time behavior of pure aggregation models.

In the following we specialize to the case of a pure aggregation model, i.e., $a \equiv 0$. We shall require G to satisfy either (IP1) or (IP3) above and the additional hypotheses

$$
\begin{gathered}
G \geq 0, \quad G \text{ has a unique maximum } \; g_0 = G(0), \\
G'(x) < 0 \text{ for } x \in (\mathrm{supp} G)^\circ \cap [0,+\infty),
\end{gathered} \tag{3.1}
$$

where A° denotes the interior of A. We shall consider several assumptions on the potential V according to different cases. The continuity equation for the density in this case reads

$$
\partial_t \rho + \partial_x\left(\rho\partial_x[G*\rho - V]\right) = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^+, \tag{3.2}
$$

and the equation for the pseudo-inverse becomes

$$
\partial_t u(z,t) = \int_0^1 G'(u(z,t)-u(\zeta,t))\;d\zeta - V'(u(z,t)) \qquad \text{in } [0,1]\times\mathbb{R}^+, \tag{3.3}
$$

In this case the above analysis allows for measure-valued solutions ρ of (3.2) due to the absence of a diffusion term, e.g. Dirac δ-distributions. In particular one expects a concentration to such measures in the long time limit, and we will show below that this is indeed true under reasonable assumptions.

For the analysis below we define the Radon measure δ_γ via

$$\langle \delta_\gamma, \varphi \rangle := \varphi(\gamma), \qquad \forall\, \varphi \in C_b^0(\mathbb{R}), \tag{3.4}$$

where $C_b^0(\mathbb{R})$ is the space of continuous and bounded functions on $\mathbb{R}$. In spatial dimension one the distribution function corresponding to δ_γ is given by

$$R_\gamma := \begin{cases} 0 & \text{if } x \le \gamma \\ 1 & \text{if } x > \gamma. \end{cases}$$

The corresponding pseudo-inverse function is then the constant function $v \equiv \gamma$. Hence, asymptotic concentration of the density ρ to a Dirac-δ is equivalent to convergence of the pseudo-inverse to a constant state. We conclude this subsection by recalling the entropy dissipation identity for equation (3.2), given by

$$\begin{aligned} &\int_{\mathbb{R}} \left(V - \frac{1}{2} G * \rho(t) \right) d\rho(t)(x) + \int_0^t \int_{\mathbb{R}} |\partial_z (V - G * \rho(\tau))|^2 d\rho(\tau)(x) d\tau \\ &= \int_{\mathbb{R}} \left(V - \frac{1}{2} G * \rho(0) \right) d\rho(0)(x), \end{aligned} \tag{3.5}$$

which coincides with (2.13) in case $a \equiv 0$.

3.1. **Stationary solutions.** We first investigate possible stationary states. As noticed above, a Dirac-δ is a good candidate, and in the following we verify that such a stationary solution indeed exists (with location determined by the potential):

Proposition 3.1 (Existence of Stationary States)**.** *Let $\gamma \in \mathbb{R}$ be such that $V'(\gamma) = 0$. Then $v \equiv \gamma$ (respectively $\rho = \delta_\gamma$) is a stationary solution of* (3.3) *(respectively* (3.2)*).*

Proof. By inserting $v \equiv \gamma$ into the right-hand side of (3.3) we obtain

$$\int_0^1 G'(v(z) - v(\zeta))\, d\zeta - V'(v(z)) = \int_0^1 G'(\gamma - \gamma)\, d\zeta - V'(\gamma) = G'(0) - V'(\gamma)$$

and since $G'(0) = 0$ and $V'(\gamma) = 0$, the last term vanishes. Hence, v is a stationary solution. $\square$

Remark 3.2. Note that if V has a unique stationary point, then Proposition 3.1 describes only a single stationary solution, while in the case $V \equiv 0$ each constant function v (and therefore each Dirac δ) is a stationary state. On the other hand, the center of mass is conserved by the evolution in the latter case, and therefore the only reasonable stationary state to consider is the one with the same center of mass as the initial value, since no other constant function v could be a reasonable limit.

Under certain extra assumptions on G and V it is possible to construct nontrivial stationary solutions to (3.2) which are linear combinations of delta distributions. We remark that the following result is a generalization of a similar existence result proven in [25], where the authors prove existence of nontrivial singular equilibria for equation (3.2) in the special case $G(x) = |x|^\gamma$, $1 < \gamma \le 4$ and $V(x) = -|x|^2/2$.

Proposition 3.3 (Nontrivial equilibria)**.** *Suppose there exists $x_0 \in (0, +\infty)$ such that*

$$G'(2x_0) = 2V'(x_0). \tag{3.6}$$

Then, the measure

$$\rho^\infty := \frac{1}{2}\left[\delta_{-x_0} + \delta_{x_0}\right]$$

is a (distributional) stationary solution to (3.2).

Proof. In order to prove the above assertion we have to check that

$$0 = \int_{\mathbb{R}} \phi'(x)\left[G' * \rho^\infty(x) - V'(x)\right] d\rho^\infty(x), \tag{3.7}$$

for any test function ϕ. By definition of ρ^∞, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \phi'(x)\left[G' * \rho^\infty(x) - V'(x)\right] d\rho^\infty(x) \\
&= \frac{1}{2}\int_{\mathbb{R}} \phi'(x)\left[G'(x - x_0) + G'(x + x_0)\right] d\rho^\infty(x) \\
&- \frac{1}{2}\left[\phi'(-x_0)V'(-x_0) + \phi'(x_0)V'(x_0)\right] \\
&= \phi'(-x_0)\left[\frac{1}{4}\left(G'(-2x_0) + G'(0)\right) - \frac{1}{2}V'(-x_0)\right] \\
&+ \phi'(x_0)\left[\frac{1}{4}\left(G'(2x_0) + G'(0)\right) - \frac{1}{2}V'(x_0)\right]
\end{aligned}$$

and the above expression vanishes because of the symmetry of G and V and of the condition $G'(0) = 0$ and in view of (3.6). ☐

Remark 3.4. We remark that the results stated in Propositions 3.1 and 3.3 require neither any convexity assumption on V nor the fact that the stationary point 0 is the unique maximizer for G. If the latter is satisfied, then it is clear from (3.6) that the nontrivial singular states found in Proposition 3.3 cannot exist when $V(x)$ is increasing for positive x, which is the case when e. g. V is uniformly convex. This fact suggests that such nontrivial states exist when the external potential V produces a repulsive drift, which is unlikely in the applications. On the other hand, a nonlocal interaction kernel with changing sign in its second derivative still allows the existence of such stationary states even when V is uniformly convex (provided that (3.6) is satisfied). An interesting situation occurs when $V \equiv 0$ and G has compact support. In this case, there exist infinitely many stationary states of the previous form, corresponding to a point x_0 such that

$$x_0 > \frac{\eta}{2}, \quad \eta = \sup_{x \in \mathrm{supp}(G)} |x|.$$

As pointed out at the end of the previous remark, stationary solutions different from the trivial one (i. e. the Dirac δ centered at the center of mass) may exists in case $V \equiv 0$ and G compactly supported. In what follows we show that the assumption of compact support for G is necessary in order to detect non trivial stationary states if G has only one stationary point, whereas infinitely many stationary states can be produced when G has compact support.

Theorem 3.5 (Uniqueness of Stationary States for Infinite Range)**.** *Let $G \in L^1(\mathbb{R})$ be such that* $\mathrm{supp}G = \mathbb{R}$ *(in addition to standard properties of G assumed above), and let $V \equiv 0$. Then each stationary solution of (3.3) is of the form $v \equiv \gamma$ for $\gamma \in \mathbb{R}$.*

Proof. Let $q = \sup_{z\in[0,1]} v(z) = v(1)$, and first assume $q < \infty$. The function v is monotonically non-decreasing, and thus, for each $\epsilon > 0$ we can find $\delta > 0$ such that

$$v(z) > q - \epsilon. \qquad \text{if } z > 1 - \delta.$$

Hence,

$$0 = \int_0^1 G'(v(z) - v(\zeta))\ d\zeta = \int_0^{1-\delta} G'(v(z) - v(\zeta))\ d\zeta + \int_{1-\delta}^1 G'(v(z) - v(\zeta))\ d\zeta$$

As $\epsilon \to 0$, we obtain the limiting inequality

$$0 \leq \int_0^1 G'(q - v(\zeta))\ d\zeta, \tag{3.8}$$

but since $G'(q - v(\zeta)) > 0$ for $v(\zeta) > q$, then (3.8) can only be true for $v \equiv q$.

If $q = \infty$, then there exists $r \in \mathbb{R}$ such that the original measure μ satisfies $G' * \mu = 0$ in (r, ∞). Consequently, $G * \mu$ is constant in (r, ∞). On the other hand, since $G \in L^1(\mathbb{R})$ and μ is a probability measure, we obtain that $G * \mu \in L^1(\mathbb{R})$ and therefore the constant value in (r, ∞) can only be zero. Hence, $G * \mu = 0$ in (r, ∞), which is only possible for $\mu \equiv 0$ due to the positivity of G, but this contradicts $\mu(\mathbb{R}) = 1$. $\square$

If on the other hand the interaction range is finite (i.e., the support of the kernel is compact) and there is no external potential, then we can immediately construct an infinite number of different stationary states:

Theorem 3.6 (Infinite number of Stationary States for Finite Range)**.** *Let G be such that* $\mathrm{supp}G = [-\eta, \eta]$ *and let* $V \equiv 0$. *Then, for each* $N \in \mathbb{N}$ *and each* $(\gamma_j)_{j=1,\dots,N} \in \mathbb{R}^N$ *such that* $\gamma_j + \eta < \gamma_{j+1}$, *the function*

$$v^N(z) := \begin{cases} \gamma_j & \text{if } \frac{j-1}{N} \leq z < \frac{j}{N} \\ \gamma_N & \text{if } z = 1 \end{cases} \tag{3.9}$$

is a stationary solution of (3.3).

Proof. Let $\frac{k-1}{N} \leq z < \frac{k}{N}$. Then,

$$\int_0^1 G'(v(z) - v(\zeta))\ d\zeta = \frac{1}{N} \sum_{j=1}^N G'(\gamma_k - \gamma_j)$$

and $G'(\gamma_k - \gamma_j) = 0$ either since $G'(0) = 0$ or since $|\gamma_k - \gamma_j| > \eta$. $\square$

To conclude this subsection, we prove that we can characterize the compactly supported stationary states also in case G has a finite range. More precisely, in terms of the pseudo inverse variable, a function v is a stationary solution to (3.3) with $V \equiv 0$ if and only if v is piecewise constant.

Theorem 3.7 (Characterization of steady states with finite range)**.** *Let G be such that (3.1) is satisfied and such that* $\mathrm{supp}G = [-\eta, \eta]$, *and let* $V \equiv 0$. *Then, all bounded stationary solutions to (3.3) are of the form (3.9) with* $|\gamma_i - \gamma_j| > \eta$ *for all* i, j.

Proof. The proof uses the same strategy of the proof of Theorem 3.5. Let once again $q = \sup_{z\in[0,1]} v(z)$. With similar arguments as in Theorem 3.5, we deduce

$$0 \leq \int_0^1 G'(q - v(\zeta))\ d\zeta.$$

Now, $G'(q - v(\zeta)) \geq 0$ may occur either for $v(\zeta) \geq q$ or for $v(\zeta) \leq q - \eta$. Let

$$\bar{\zeta} = \sup\{\zeta \in [0,1) \mid v(\zeta) \leq q - \eta\}.$$

Then, we have

$$\int_{\bar{\zeta}}^{1} G'(q - v(\zeta))\, d\zeta = \int_{0}^{1} G'(q - v(\zeta))\, d\zeta - \int_{0}^{\bar{\zeta}} G'(q - v(\zeta))\, d\zeta \geq 0.$$

Now, since $G'(q - v(\zeta)) \leq 0$ for $\zeta \in [\bar{\zeta}, 1]$, we can deduce that $v(\zeta) \equiv q$ over the interval $[\bar{\zeta}, 1]$. We can now repeat the same argument on the interval $[0, \bar{\zeta}]$ in order to prove that v is constant on an interval $[\tilde{\zeta}, \bar{\zeta}]$. The proof can be completed by iteration, where the finiteness of the number of steps is guaranteed by the condition $q - v(\bar{\zeta}) \geq \eta$ and by the fact that v is bounded. The last assertion of the Theorem is a consequence of Theorem 3.8. □

3.2. **Self–Similar solutions.** In this subsection we shall analyze the existence of so called self–similar solutions of the form

$$\rho(x,t) = \frac{1}{2}\delta_{-y(t)} + \frac{1}{2}\delta_{y(t)} \tag{3.10}$$

of the equation (3.2) under the standard assumption that $V' \geq 0$ on $[0, +\infty)$ and G satisfies (3.1). In terms of the pseudo inverse equation (3.3), a solution of the form (3.10) can be expressed in terms of a function $t \to y(t)$ solving the ordinary differential equation

$$\dot{y}(t) = \frac{1}{2}G'(2y(t)) - V'(y(t)). \tag{3.11}$$

The equation (3.11) can be easily recovered by substituting

$$u(z,t) = \begin{cases} -y(t) & \text{if } \ z \in \left[0, \frac{1}{2}\right) \\ y(t) & \text{if } \ z \in \left[\frac{1}{2}, 1\right] \end{cases}$$

in the equation (3.3) and by choosing $z \in (1/2, 1]$, whereas the equation will be automatically satisfied for $z \in [0, 1/2]$. As can be easily seen, the stationary solutions found in Theorem 3.3 are obtained by means of constant solutions to (3.11), since condition (3.6) is nothing but the stationary equation corresponding to (3.11). Suppose now that (3.6) is not satisfied by the initial datum, more precisely suppose that $y(0) = y_0 > 0$ and

$$G'(2y_0) - 2V'(y_0) < 0$$

(we recall that $G'(2y_0) - 2V'(y_0) \leq 0$ for all $y_0 > 0$ because of the assumptions on G and V). Then we can explicitly compute the solution to (3.11) by direct integration to obtain

$$y(t) = F^{-1}(-t), \quad F(x) = \int_{y_0}^{x} \frac{1}{V'(\xi) - \frac{1}{2}G'(2\xi)} d\xi. \tag{3.12}$$

Clearly, the solution $t \to y(t)$ is decreasing and therefore it admits a limit for large time. Let then $\bar{y} := \lim_{t \to +\infty} y(t)$ and suppose that $\bar{y} \neq 0$. Then we can write

$$+\infty = \int_{\bar{y}}^{y_0} \frac{1}{V'(\xi) - \frac{1}{2}G'(2\xi)} d\xi$$

which is a contradiction because the integrand above is bounded away from $\xi = 0$. Therefore $y(t)$ tends to zero for large times. We have thus proven that the equation (3.2) exhibits self–similar solutions of the form (3.10) with $y(t) \to 0$ as $t \to +\infty$.

Depending on the behavior of G' and V' at zero, $y(t)$ attains the value zero at finite or infinite time. More precisely, let

$$t^\infty := \sup\{t > 0 \mid y(t) > 0\}.$$

Then, we have

$$t^\infty = \int_0^{y_0} \frac{1}{V'(\xi) - \frac{1}{2}G'(2\xi)} d\xi.$$

We summarize all the previous computations in the following Theorem.

Theorem 3.8. *Suppose $V'' \geq 0$ and G satisfies (3.1). Let*

$$G'(2y_0) - 2V'(y_0) < 0$$

and let $y(t) > 0$ be given by (3.12). Then, the measure

$$\rho(x,t) = \frac{1}{2}\delta_{-y(t)} + \frac{1}{2}\delta_{y(t)}$$

solves the equation (3.2) in weak sense. Moreover, given the (eventually infinite) time

$$t^\infty = \int_0^{y_0} \frac{1}{V'(\xi) - \frac{1}{2}G'(2\xi)} d\xi,$$

$y(t)$ is strictly increasing on $[0, t^\infty)$ and $\lim_{t \to t^\infty} y(t) = 0$.

3.3. **Attractors of the evolution.** We expect the convergence of arbitrary solutions to stationary solutions (whose existence is guaranteed from the results above) for large times. In a general setup, this is true at least in a weak sense for subsequences as we shall show now:

Theorem 3.9. *Suppose $V \geq 0$ and suppose that V satisfies $V'(x) \geq 0$ as $x \geq 0$ and $V(-x) = V(x)$. Let u be the solution of (3.3) and $\mu(t)$ the probability measure corresponding to $u(.,t)$ with compactly supported initial value $\mu_0 = \mu(0)$. Then, there exists a sequence $t_k \to \infty$ and a probability measure μ^∞ such that $\mu(t) \to \mu^\infty$ narrowly in $\mathcal{P}(\mathbb{R})$, and each weak limit of a subsequence of $\mu(t)$ is a stationary solution of (3.2) (respectively the pseudo-inverse of the distribution function is a stationary solution of (3.3)).*

Proof. From the weak formulation (2.3) applied to the test function $\phi(x) = x^4$, by observing that

$$\int_{\mathbb{R}} x^3 G' * \mu(t) d\mu(t) \leq 0,$$

we easily obtain

$$\int_{\mathbb{R}} x^4 d\mu(t) \leq \int_{\mathbb{R}} x^4 d\mu(0),$$

which implies that $\mu(t)$ is a a tight family in $\mathcal{P}(\mathbb{R})$ (i. e. $\rho(t)$ is relatively compact with respect to the narrow convergence topology). The last assertion easily follows by observing

$$\int_{x \geq R} (1 + x^2)\rho(x,t)dx = \int_{x \geq R} \frac{(1+x^2)^2}{1+x^2}\rho(x,t)dx \leq C\frac{1}{1+R^2}.$$

Hence, we can extract a diverging subsequence denoted by t_k such that $\mu(t_k)$ converges to some limit μ^∞ narrowly in $\mathcal{P}(\mathbb{R})$. Due to the energy dissipation relation (3.5) we know that

$$\int_0^\infty \langle \mu(t), |\partial_x (G * \mu(t) - V)|^2 \rangle dt < \infty.$$

Hence (eventually up to subsequences of t_k),

$$\langle \mu(t_k), |\partial_x(G * \mu(t_k) - V)|^2 \rangle \to 0.$$

Since the above functional is lower semicontinuous with respect to the 2–Wasserstein distance, the limit μ^∞ must be a stationary solution. ☐

For infinite range interactions, the results can be strengthened, since the only stationary solution is a Dirac-δ at the center of mass

Corollary 3.10. *Let the assumptions of Theorem 3.9 be satisfied and let G be as in Theorem 3.5. Then $\mu(t) \to \delta_{CM}$ narrowly in $\mathcal{P}(\mathbb{R})$ as $t \to \infty$.*

3.4. **Moment estimates and convergence rates to singular equilibria.** In this section we focus on contraction and stability properties of solutions of (3.2) with respect to the Wasserstein distances. In order to perform this task, we shall deal with L^p estimates in terms of solutions to the pseudo–inverse equation (3.3). First we shall prove that the solutions to (3.3) are stable in L^p. Such a stability turns out to be a strict contraction in case V has a unique minimum at zero, and this implies that the p–moments of the solution ρ to equation (3.2) are decreasing in time. The convolution kernel G will be supposed to satisfy (IP1) or (IP3) and (3.1). The estimates performed in this section and in the following one are natural generalizations of the estimates in [25].

Throughout this section, $\mu(t)$ will denote the solution of (3.2), while $u(t)$ will denote the corresponding pseudo–inverse satisfying (3.3). The L^p estimates on $u(t)$ can be easily translated in terms of W_p estimates on the measure valued solution $\mu(t)$ thanks to the results in Theorems 2.9 and 2.11.

We start with the following theorem.

Theorem 3.11. *Suppose that G satisfies either (IP1) of (IP3) together with (3.1). Moreover assume that the potential V satisfies*

$$V'(u) \geq \alpha u, \qquad \forall\ u \geq 0.$$

Let $u(z,t)$ be the solution of (3.3) having initial datum $u_0 \in L^p([0,1])$. Then, for $p \in [1, +\infty]$

$$W_p(\mu(t), \delta_0) = \|u(t)\|_{L^p([0,1])} \leq e^{-\alpha t} \|u_0\|_{L^p([0,1])},$$

for all $t \geq 0$.

Proof. We formally compute the time evolution of the L^p norm of u. The following estimate can be made rigorous by smoothing the function $|\cdot|^p$ in a standard way. In the following we drop the time dependency in the notation for simplicity, and derive

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 |u(z,t)|^p dz = p\int_0^1 |u(z)|^{p-1}\mathrm{sign}(u(z))u_t dz \\
&= p\int_0^1\int_0^1 |u(z)|^{p-1}\mathrm{sign}(u(z))[G'(u(z)-u(\zeta))]dzd\zeta \\
&\ - p\int_0^1 |u(z)|^{p-1}\mathrm{sign}(u(z))V'(u(z))dz \\
&=: I_1 + I_2.
\end{aligned}$$

Due to the antisymmetry of G' and to $G'(z)$ being nonnegative for nonnegative z, using the monotonicity of the function $u \mapsto |u|^{p-1}\mathrm{sign}(u)$, we can compute the term I_1 as follows,

$$I_1 = p\int\int_{z\leq\zeta} |u(z)|^{p-1}\mathrm{sign}(u(z))[G'(u(z)-u(\zeta))]dzd\zeta +$$
$$p\int\int_{z\geq\zeta} |u(z)|^{p-1}\mathrm{sign}(u(z))[G'(u(z)-u(\zeta))]dzd\zeta$$
$$\leq p\int\int_{z\leq\zeta} |u(\zeta)|^{p-1}\mathrm{sign}(u(\zeta))[G'(u(z)-u(\zeta))]dzd\zeta +$$
$$-p\int\int_{z\geq\zeta} |u(z)|^{p-1}\mathrm{sign}(u(z))[G'(u(\zeta)-u(z))]dzd\zeta = 0.$$

The assumption on V' implies

$$I_2 \leq -\alpha p\int_0^1 |u(z)|^p dz$$

and the assertion follows with the Gronwall lemma for finite p. The case $p=\infty$ can be obtained by taking the limit $p\to+\infty$. ☐

The results in Theorem 3.11 provide exponential convergence of $\rho(t)$ in the p–Wasserstein distance towards the Dirac δ centered at zero in case $\alpha>0$. A similar situation occurs in case $V\equiv 0$, namely

$$\partial_t u = \int_0^1 G'(u(z)-u(\zeta))d\zeta. \tag{3.13}$$

As shown in a previous section, in this case the suitable stationary state $u\equiv\gamma$ is determined by the center of mass, which is invariant under the flow. We shall therefore consider initial data with zero center of mass for simplicity. Again, the behavior of the solutions depends on the range of G being finite or infinite. In case of infinite range, we shall prove that the second moment (or, equivalently the 2–Wasserstein distance to δ_0) of any compactly supported solution converges to zero for large times. The rate of convergence depends on the size of the support of initial data and on the behavior of G' near zero, as it can be seen in the statement of the following theorem. Again, the proof is reminiscent of [25].

Theorem 3.12. *Suppose G satisfies either (IP1) of (IP3) together with (3.1) and suppose $V\equiv 0$. Moreover, suppose that* $\mathrm{supp}(G)=\mathbb{R}$ *and that*

$$\lim_{x\to 0^+}\frac{G'(x)}{x^\alpha} = l < 0, \tag{3.14}$$

for some $\alpha>0$. Let $u(z,t)$ be the solution to (3.13) having initial datum $u_0\in L^\infty([0,1])$ (or, equivalently, $\mathrm{supp}(\rho(0))$ *compact). Then, there exists a constant $C>0$ depending on G and on*

$$\|u_0\|_{L^\infty([0,1])} = \max\{|\inf\mathrm{supp}(\rho(0))|, |\sup\mathrm{supp}(\rho(0))|\}$$

such that,

- $\|u(t)\|_{L^{2k}([0,1])} \leq e^{-Ct}\|u_0\|_{L^{2k}([0,1])}$, *for any positive integer k, if $\alpha=1$,*
- $\|u(t)\|_{L^2([0,1])} \leq C(1+t)^{-\frac{1}{\alpha-1}}$ *if $\alpha>1$,*
- *there exists $t^*>0$ such that $\|u(t)\|_{L^\infty([0,1])}=0$ for $t\geq t^*$ if $0<\alpha<1$.*

Proof. As in the previous theorem, by direct computation of the evolution of the L^{2k} norm of $u(t)$ we have

$$\frac{d}{dt}\int_0^1 u(z,t)^{2k}dz = 2k\int_0^1 u(z)^{2k-1}u_t dz = 2k\int_0^1\int_0^1 u(z)^{2k-1}[G'(u(z)-u(\zeta))]dzd\zeta$$
$$= 2k\int\int_{z\leq\zeta} u(z)^{2k-1}[G'(u(z)-u(\zeta))]dzd\zeta$$

$$+\,2k\int\int_{z\geq\zeta} u(z)^{2k-1}[G'(u(z)-u(\zeta))]dzd\zeta$$
$$= 2k\int\int_{z\geq\zeta}[u(z)^{2k-1}-u(\zeta)^{2k-1}][G'(u(z)-u(\zeta))]dzd\zeta.$$

Now, thanks the result in Theorem 3.11,

$$|u(z)-u(\zeta)|\leq 2\|u(t)\|_{L^\infty}\leq 2\|u(0)\|_{L^\infty},$$

and the above inequality, together with hypotheses (3.14) and (3.1), guarantees that

$$\frac{[G'(u(z)-u(\zeta))]}{[u(z)-u(\zeta)]^\alpha}\leq -L \tag{3.15}$$

for a certain $L>0$ (L depending on $\|u(0)\|_{L^\infty}$) when $z\geq\zeta$. We observe here that, in view of the hypotheses $G\in L^1$, one has to choose a suitably small constant L in (3.15) when the size of the initial support of ρ (i. e. the initial L^∞–norm of u) is very large. Therefore, at this level of the proof the infinite size of the range of G plays a key role. Due to (3.15) we have

$$\frac{d}{dt}\int_0^1 u(z,t)^{2k}dz\leq -2kL\int\int_{z\geq\zeta}[u(z)^{2k-1}-u(\zeta)^{2k-1}][u(z)-u(\zeta)]^\alpha dzd\zeta. \tag{3.16}$$

Using the conservation of the first moment, in a similar fashion as in [25], we can prove the following estimate which will be useful in the sequel,

$$\int_0^1 u(z)^{2k}dz\leq\int_0^1\int_0^1[u(z)-u(\zeta)]^2u(z)^{2k-2}dzd\zeta$$
$$\leq\frac{1}{2}\int_0^1\int_0^1[u(z)-u(\zeta)]^2[u(z)^{2k-2}+u(\zeta)^{2k-2}]dzd\zeta$$
$$\leq\frac{1}{2}\int_0^1\int_0^1[u(z)-u(\zeta)][u(z)^{2k-1}-u(\zeta)^{2k-1}]dzd\zeta, \tag{3.17}$$

We now consider the following three cases corresponding to different ranges of values of α:
Case $\alpha>1$. The inequality (3.16) when $k=1$ implies in this case

$$\frac{d}{dt}\int_0^1 u(z,t)^2dz\leq -2L\int\int_{z\geq\zeta}[u(z)-u(\zeta)]^{1+\alpha}dzd\zeta$$
$$= -L\int_0^1\int_0^1[u(z)-u(\zeta)]^{1+\alpha}dzd\zeta.$$

By Hölder inequality and by conservation of the first moment, we have

$$\frac{d}{dt}\int_0^1 u(z,t)^2 dz \leq -L\int\left[\int_{z\geq\zeta}[u(z)-u(\zeta)]^2 dz d\zeta\right]^{\frac{1+\alpha}{2}}$$
$$= -L2^{\frac{1+\alpha}{2}}\left(\int_0^1 u(z,t)^2 dz\right)^{\frac{1+\alpha}{2}}.$$

Hence, the variation of constants formula applied to the above inequality implies the assertion.

CASE $\alpha = 1$. The inequalities (3.16) and (3.17) imply

$$\frac{d}{dt}\int_0^1 u(z,t)^{2k} dz \leq -2kL\int_0^1 u(z,t)^{2k},$$

and the assertion follows with the Gronwall lemma.

CASE $0<\alpha<1$. The proof in this case can be performed in the same way as in [25, Section 6, Theorem 6.1] starting from the inequalities (3.16) and (3.17). Therefore, we shall skip the further details here. □

Notice that the existence of a solution in the case $\alpha \in (0,1)$ is ensured by Theorem 2.11. Obviously, the above result has to be interpreted in terms of convergence in p–Wasserstein distance of $\mu(t)$ to the Delta measure. In particular, in the case $0<\alpha<1$ the support of $\mu(t)$ degenerates in finite time.

Let us now focus on the case of an interaction kernel G having finite support. As in Theorem 3.6 we define

$$\mathrm{supp} G := [-\eta, \eta],$$

for some $\eta > 0$. As shown in Theorem 3.6, in this case we have infinite singular equilibria for equation (3.2) when $V \equiv 0$. The selection of the right stationary solution as a typical asymptotic state depends on the location of the support of the initial datum. In particular, we shall prove that when the initial datum has many connected components separated more than the range of action of the interaction kernel, then each connected component behaves independently from the others. In particular, each connected component will converge weakly to a Dirac δ centered at its first moment. This result is summarized in the following theorem.

Theorem 3.13. *Let ρ be the solution to (3.13), with G satisfying (3.14) for some $\alpha>0$, with a compactly supported probability measure μ_0 as initial datum. Suppose that $\mathrm{supp}\mu_0 = \bigcup_{j=1}^n [a_j, b_j]$, with $a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n$ for a certain fixed integer n. Moreover, suppose that $b_j - a_j < \eta$ for all $j \in \{1,\ldots,n\}$ and that $a_{j+1} - b_j > \eta$ for all $j \in \{1,\ldots,n-1\}$. Let*

$$m_j := \int_{a_j}^{b_j} d\mu_0(x), \qquad \gamma_j := \int_{a_j}^{b_j} x d\mu_0(x), \quad j = 1,\ldots,n.$$

Then,

$$W_p\left(\mu(t), \sum_{j=1}^n m_j \delta_{\gamma_j}\right) \to 0,$$

for all $p \in [2,+\infty]$.

Proof. Step 1. Let us first consider the case of one single connected component, namely supp$\rho_0 = [a, b]$ with $b-a < \eta$. Then, in terms of the pseudo inverse variable u it can be easily proven that $u(0,t)$ and $u(1,t)$ are nondecreasing and nonincreasing with respect to t respectively. In order to see that, we compute

$$u_t(1,t) = \int G'(u(1) - u(\zeta))d\zeta \leq 0,$$

due to $G'(z) \leq 0$ for $z \geq 0$. The assertion at $z = 0$ can be proven similarly. As a consequence of that, we have

$$u(1,t) - u(0,t) < \eta$$

for all $t \geq 0$. Therefore, we can derive the inequality (3.15) due to assumption (3.1) and repeat the proof of Theorem 3.12 in order to get the desired assertion.

Step 2. In order to simplify the notation, we shall perform the proof in the case $n = 2$. The general case $n > 2$ does not bring any further significant difficulty to the problem and it can be performed by an analogous argument. Let us then suppose that supp$\rho_0 = [a, b] \cup [c, d]$, with $b - a < \eta$, $d - c < \eta$, $c - b > \eta$. Let

$$m_1 = \int_a^b d\rho_0(x), \quad m_2 = \int_c^d d\rho_0(x) = 1 - m_1, \quad \gamma_1 = \int_a^b x d\rho_0(x), \quad \gamma_2 = \int_c^d x d\rho_0(x).$$

We claim that the solution can be expressed in terms of the pseudo inverse variable u as follows:

$$u_t = \int_0^{m_1} G'(u(z) - u(\zeta))d\zeta \qquad \text{if } \ 0 \leq z < m_1 \tag{3.18}$$

$$u_t = \int_{m_1}^1 G'(u(z) - u(\zeta))d\zeta \qquad \text{if } \ m_1 \leq z \leq 1. \tag{3.19}$$

More precisely, assuming that u^0 be the pseudo inverse of the primitive of ρ_0, we claim that the solution u to (3.13) with initial datum u^0 is given by $u_1(z)$ for $z < m_1$ and by $u_2(z)$ for $z \geq m_1$, where u_1 and u_2 solve (3.18) and (3.19) respectively, with initial data $u_1^0 = u^0|_{[0,m_1)}$ and $u_2^0 = u^0|_{[m_1,1]}$ respectively. In order to prove such an assertion, we observe that the aforementioned u actually solves the equation (3.13) with initial datum u_0, because it can be easily proven that $\lim_{z \searrow m_1} u(z,t) - \lim_{z \nearrow m_1} u(z,t) > c - b$ for all $t \geq 0$ (because $G'(z) \leq 0$ when $z \geq 0$). Therefore, the assertion follows by uniqueness of the solution.

Finally, since u_1 and u_2 behave like two solutions to (3.13) (with masses m_1 and $1 - m_1$ respectively) with initial data having one single component in its support, we can apply step one and the proof is complete. ∎

Remark 3.14. Explicit rates of convergence in the previous Theorem can be derived as in Theorem 3.12, depending on the value of the constant α.

Remark 3.15. Clearly, an open problem is determining the asymptotic behavior when G has finite range $[-\eta, \eta]$ and when the initial support does not satisfy the hypotheses of the previous theorem. For instance, it is not clear whether a solution with initial support exceeding the value η in its diameter will converge to the Dirac δ centered at the center of mass or it will develop many peaks.

4. **Small diffusion models.** In the following we consider the case of a small nonlinear diffusion. More precisely, we assume $a(\rho) = \epsilon\rho^{m-1}$ for $0 < \epsilon \ll 1$ and investigate the asymptotic $\epsilon \to 0$ in order to achieve existence of stationary solutions. For simplicity we also assume that there is no external potential, i.e., $V \equiv 0$. The interaction kernel will be supposed to satisfy (IP) plus the additional assumptions

- $G \in C^3(I)$ for some neighborhood I of zero.
- G is decreasing on $\mathbb{R}_+$ and G has a unique maximum at zero.
- $G''(0) < 0$.

Equation (2.10) in this case reads

$$\partial_t u(z,t) = -\frac{m-1}{m}\partial_z\left(\frac{\epsilon}{(\partial_z u(z,t))^m}\right) + \int_0^1 G'(u(z,t) - u(\zeta,t))\, d\zeta, \tag{4.1}$$

in $[0,1] \times \mathbb{R}^+$.

4.1. **Formal asymptotic expansion.** For $\epsilon = 0$ we have studied the stationary states of this equation above, and a possible choice is always the constant state equal to the center of mass CM. Motivated by this limiting solution we study a formal expansion for the pseudo-inverse of the form

$$u = CM + \epsilon^\nu u^1 + o(\epsilon^\nu),$$

with $\int_0^1 u^1\, dz = 0$ (reflecting the conservation of the integral of u). Then we obtain

$$\epsilon^\nu \partial_t u^1(z,t) = -\frac{m-1}{m}\partial_z\left(\frac{\epsilon^{1-m\nu}}{(\partial_z u^1(z,t))^m}\right) + \epsilon^\nu \int_0^1 G''(0)(u^1(z,t) - u^1(\zeta,t))\, d\zeta + o(\epsilon^\nu).$$

which determines the natural choice for the exponent $\nu = \frac{1}{m+1}$. Since the integral term evaluates to

$$\begin{aligned}\int_0^1 G''(0)(u^1(z,t) - u^1(\zeta,t))\, d\zeta &= G''(0)\int_0^1 u^1(z,t)\, d\zeta - G''(0)\int_0^1 u^1(\zeta,t)\, d\zeta \\ &= G''(0)u^1(z,t),\end{aligned}$$

the first-order expansion is determined by the local equation

$$\partial_t u^1 = -\frac{m-1}{m}\partial_z\left(\frac{1}{(\partial_z u^1)^m}\right) + cu^1. \tag{4.2}$$

Since the factor $c := G''(0)$ is nonpositive, the last term adds a decaying mode, thus playing the role of a confining potential in a nonlinear Fokker-Planck equation.

Now we can take a closer look on stationary solutions of (4.2), which solve

$$-\frac{m-1}{m}\partial_z\left(\frac{1}{(\partial_z v^1)^m}\right) = |c|u^1.$$

After multiplying by $\partial_z v^1$, we can integrate with respect to z to obtain

$$-\frac{1}{(\partial_z v^1)^{m-1}} = \frac{|c|}{2}(v^1)^2 - \gamma$$

for $m > 1$, with some integration constant γ. Since $\frac{1}{\partial_z v^1} \to 0$ as $z \to 0$ or $z \to 1$, the solution satisfies

$$\frac{1}{\partial_z v^1} = \left[\left(\gamma - \frac{|c|}{2}(v^1)^2\right)_+\right]^{1/(m-1)},$$

with constant γ determined from the mass.

We can relate v again to stationary solution of (1.1), to leading order we have $\rho = \frac{1}{\epsilon^\nu \partial_z v} \approx \frac{1}{\partial_z v^1}$ and $x \approx CM + \epsilon^\nu v^1$. Hence, a stationary solution is asymptotically determined as

$$\rho^\infty(x) = \epsilon^{-\nu}\left[\left(\gamma - \frac{|c|}{2}\epsilon^{-2\nu}(x - CM)^2\right)_+\right]^{1/(m-1)}$$

Consequently we expect a stationary solution with support of order ϵ^ν. In a similar way we can construct asymptotic expansions around stationary solutions with multiple peaks (as existing in the case of a kernel with compact support). Thus, to leading order it seems that the stationary solutions of (1.1) for small ϵ are of the same structure as the ones for $\epsilon = 0$, but the Dirac δ distributions are changed to finite peaks of height $\epsilon^{-\nu}$.

4.2. **Existence of stationary states with compact support.** For the particular case $m = 2$ we can make the above reasoning rigorous to some extent, i.e., we can prove the existence of stationary solutions with compact support. We start from the stationary version of (4.1)

$$0 = \frac{1}{2}\partial_z\left(\frac{\epsilon}{(\partial_z u(z))^2}\right) - \int_0^1 G'(u(z) - u(\zeta))\, d\zeta \qquad \text{in } [0,1]. \tag{4.3}$$

Accordingly to the previous formal asymptotic expansion, we make the ansatz

$$u = CM + \delta v, \qquad \delta = \epsilon^{1/3}.$$

We then look for v solving

$$0 = \frac{1}{2}\partial_z\left(\frac{\delta}{(\partial_z v)^2}\right) - \int_0^1 G'(\delta(v(z) - v(\zeta))d\zeta.$$

This equation can be rewritten as

$$-\frac{\delta^2\partial_{zz}v(z)}{(\partial_z v(z))^2} = \delta\int_0^1 G'(\delta(v(z) - v(\zeta)))\, d\zeta\partial_z v(z) = \partial_z\int_0^1 G(\delta(v(z) - v(\zeta)))\, d\zeta,$$

and hence, we can integrate with respect to z to obtain

$$\frac{\delta^2}{\partial_z v(z)} = \int_0^1 G(\delta(v(z) - v(\zeta)))\, d\zeta + \alpha \tag{4.4}$$

for some integration constant α. By substituting $z = 1$ into the above identity we get

$$\alpha = -\int_0^1 G(\delta(v(1) - v(\zeta)))\, d\zeta,$$

and the analogous condition is satisfied at $z = 0$ if we look for an antisymmetric v, i. e. $v(1 - z) = -v(z)$, which implies that the corresponding density is even. We recall that the term $1/v_z$ vanishes at $z = 0, 1$ if the corresponding density is compactly supported and continuous. Now we can multiply (4.4) by $\delta\partial_z v(z)$ and integrate again to deduce

$$\delta^3 z = \int_0^1 H(\delta(v(z) - v(\zeta)))\, d\zeta + \alpha\delta v(z) + \beta, \tag{4.5}$$

where β is another integration constant and $H' = G$. Without restriction of generality we assume that $H(0) = 0$. Since H is an odd kernel, we obtain by integration with respect to z that

$$\frac{\delta^3}{2} = \beta.$$

Equation (4.5) can be rewritten as

$$0 = \frac{1}{2} - z + \delta^{-3}\int_0^1 \left[H(\delta(v(z) - v(\zeta))) - \delta v(z) G(\delta(v(1) - v(\zeta)))\right] d\zeta. \tag{4.6}$$

The expression (4.6) can be viewed as a function equation of the form $F(v, \delta) = 0$, where $\delta > 0$ is a parameter and where the map F is defined as

$$F(v,\delta) := \begin{cases} \frac{1}{2} - z + \delta^{-3}\int_0^1 \left[H(\delta(v(z) - v(\zeta))) - \delta v(z) G(\delta(v(1) - v(\zeta)))\right] d\zeta & \\ & \text{if } \ \delta \neq 0 \\ \frac{1}{2} - z + \frac{1}{6} G''(0)[v(z)^3 - 3v(z)v(1)^2] & \\ & \text{if } \ \delta = 0. \end{cases}$$

If we assume (so far) that v lives in the functional space

$$\mathcal{X} := \left\{ v \in L^\infty[0,1], \quad v \text{ increasing}, \quad \int_0^1 v(z)dz = 0, \quad v(1-z) = -v(z) \right\},$$

we can easily recover the above expression for $\delta = 0$ by the following Taylor expansion of the integrand in (4.6). More precisely (recalling $H(0) = 0$, $H' = G$, $G'(0) = 0$), we have

$$\begin{aligned} &\delta^{-3}\int_0^1 \left[H(\delta(v(z) - v(\zeta))) - \delta v(z) G(\delta(v(1) - v(\zeta)))\right] d\zeta \\ &\quad = \delta^{-3}\int_0^1 \bigg[\delta G(0)(v(z) - v(\zeta)) + \frac{\delta^3}{6} G''(0)(v(z) - v(\zeta))^3 - G(0)\delta v(z) \\ &\quad\quad - \frac{\delta^3}{2} G''(0) v(z)(v(1) - v(\zeta))^2 + \delta^4 R(\delta, v)\bigg] d\zeta \end{aligned} \tag{4.7}$$

The assumption $v(1-\zeta) = -v(\zeta)$ (which corresponds to $\rho(-x) = \rho(x)$ at the level of the density) and further simple calculations imply that the extension to $\delta = 0$ is identified by the term in the above definition. Moreover, the boundedness of v ensures the remainder R above is uniformly bounded and the extension is therefore continuous. For further use, we observe that $\mathcal{X}$ can be identified with the space

$$\mathcal{X}^* := \{ v \in L^\infty[1/2, 1], \quad v \text{ increasing}, \quad v(1/2) = 0 \},$$

via antisymmetric extension to the interval $[0, 1/2]$.

Our strategy to prove existence of a certain v for small $\delta > 0$ is to use the Implicit Function Theorem (cf. [14, Theorem 15.1]). For $\delta = 0$ the functional equation (4.6) reads

$$\frac{6\left(z - \frac{1}{2}\right)}{G''(0)} = v(z)^3 - 3v(z)v(1)^2. \tag{4.8}$$

It is easily seen that the unique v_0 satisfying (4.8) is the pseudo inverse of the primitive of the density

$$\rho_0(x) = \frac{|G''(0)|}{2}\left(x^2 - |G''(0)|^{-\frac{2}{3}}\right)_+$$

and we have $v_0(1) = -\left(\frac{2G''(0)}{3}\right)^{-1/3}$. The uniqueness of such a v_0 easily follows by recalling that (4.8) is the stationary pseudo–inverse equation satisfied in the case of nonlinear Fokker–Planck equation with quadratic potential, which has a unique solution (a Barenblatt–type profile).

For fixed $\delta \geq 0$ we shall regard the mapping $F(\cdot,\delta)$ as an operator defined on the domain $\mathcal{X}^*_{1/2}$ onto $\mathcal{X}^*_1$, where we have used the notation

$$\mathcal{X}^*_\alpha := \left\{ v \in \mathcal{X}^* \mid \sup_{1/2 \leq z \leq 1} \frac{v(1)-v(z)}{(1-z)^\alpha} < +\infty \right\}$$

$$|||v|||_\alpha := \|v\|_{L^\infty[0,1]} + \sup_{1/2 \leq z \leq 1} \frac{v(1)-v(z)}{(1-z)^\alpha}.$$

It is easy to check that the extension of the map $F(\cdot,\delta) : \mathcal{X}^*_{1/2} \to \mathcal{X}^*_1$ to $\delta = 0$ is still continuous. In order to see that, one can express the remainder in (4.7) by means of mean value formulas and obtain a uniform bound for R in the $|||\cdot|||_1$ norm. We have the following lemmas.

Lemma 4.1. *The solution v_0 to the functional equation $F(v_0, 0) = 0$ belongs to the space $\mathcal{X}^*_{1/2}$ and the ratio*

$$\frac{v_0(1) - v_0(z)}{(1-z)^{1/2}}$$

is strictly positive on $[1/2, 1]$.

Proof. Let $F_0(x) := \int_{-\infty}^x \rho_0(y)dy$. Since F_0 (restricted to its support) is the inverse of v_0, it suffices to prove

$$\lim_{x \to v_0(1)^-} \frac{F_0(x) - 1}{(x - v_0(1))^2} = l > 0.$$

In order to prove that, we use De L'Hospital rule twice to get

$$\lim_{x \to v_0(1)^-} \frac{F_0(x) - 1}{(x - v_0(1))^2} = \lim_{x \to v_0(1)^-} \frac{1}{2}\rho_0'(x) = \frac{|G''(0)| v_0(1)}{2} > 0.$$

$\square$

Lemma 4.2. *For any fixed $\delta \geq 0$, $F(\cdot,\delta) : \mathcal{X}^*_{1/2} \to \mathcal{X}^*_1$ is a bounded operator.*

Proof. Let $v \in \mathcal{X}^*_{1/2}$ and let $h(z) := F(v,\delta)$. From the definition of F it is clear that the L^∞ norm of h can be bounded by a constant depending on the L^∞ norm of v. We now compute

$$\begin{aligned} h(z) - h(1) = 1 - z + \delta^{-3} \int_0^1 & [H(\delta(v(z) - v(\zeta))) - H(\delta(v(1) - v(\zeta))) \\ & -\delta G(\delta(v(1) - v(\zeta)))v(z) + \delta G(\delta(v(1) - v(\zeta)))v(1)]\, d\zeta. \end{aligned}$$

By means of the first order Taylor expansion of H centered at $\delta(v(1) - v(\zeta))$ in the integral above, we obtain

$$h(z) - h(1) = 1 - z + \delta^{-1}(v(z) - v(1))^2 \int_0^1 R(z,\zeta,\delta)d\zeta,$$

for a certain bounded function R. By dividing the above relation by $\zeta - 1$ and thanks to the assumption on v we obtain the desired control of the difference quotient of h at the point $z = 1$. $\square$

We now pass to study the partial (functional) derivative of F with respect to v at the point $(v_0, 0)$. For all $w \in \mathcal{X}^*_{1/2}$ we define

$$g(z) := \frac{\partial F}{\partial v}(v_0, 0)[w](z) = \frac{G''(0)}{2}\left[v(z)^2 w(z) - v_0(1)^2 w(z) - 2v_0(z)v_0(1)w(1)\right].$$

Lemma 4.3 (Boundedness of the partial derivative)**.** *Let $|||w|||_{1/2} \leq 1$. Then, there exists a fixed constant $A > 0$ such that $|||g|||_1 \leq A$.*

Proof. It is clear that $\|g\|_{L^\infty}$ can be uniformly bounded. We then compute

$$g(1) = -G''(0)v_0(1)^2 w(1).$$

We now evaluate the difference $g(z) - g(1)$ as follows,

$$\begin{aligned} g(z) - g(1) &= \frac{G''(0)}{2}\left[w(z)(v_0^2(z) - v_0^2(1)) - 2v_0(1)w(1)(v_0(z) - v_0(1))\right] \\ &= \frac{G''(0)}{2}(v_0(z) - v_0(1))\left[w(z)(v_0(z) + v_0(1)) - 2v_0(1)w(1)\right] \\ &= \frac{G''(0)}{2}(v_0(z) - v_0(1)) \times \\ &\quad \times \left[(w(z) - w(1))\,(v_0(z) + v_0(1)) + w(1)\,(v_0(z) - v_0(1))\right]. \end{aligned}$$

By dividing the above relation by $z - 1$ and by using the hypotheses on w and the result in Lemma 4.1, we obtained the desired assertion. ☐

Finally, we prove that the inverse of $\frac{\partial F}{\partial v}(v_0, 0)$ is also bounded, namely

$$g \in \mathcal{X}^*_1 \mapsto w := \left(\frac{\partial F}{\partial v}(v_0, 0)\right)^{-1}[g] \in \mathcal{X}^*_{1/2}$$

$$w(z) = \left(\frac{2}{G''(0)}\right)\frac{g(z) - \frac{v_0(z)}{v_0(1)}g(1)}{v_0^2(z) - v_0^2(1)}$$

Lemma 4.4 (Boundedness of the inverse)**.** *Let $|||g|||_1 \leq 1$. Then, there exists a fixed constant $A > 0$ such that $|||w|||_{1/2} \leq A$.*

Proof. A simple but tedious calculation gives

$$\begin{aligned} w(z) - w(1) &= \frac{2g(z)v_0^3(1) - 2v_0(z)g(1) + g(1)v_0(1)v_0^2(z) - g(1)v_0^3(1)}{G''(0)v_0^3(1)(v_0^2(z) - v_0^2(1))} \\ &= \frac{2v_0^3(1)(g(z) - g(1)) + g(1)v_0(1)(v_0(z) - v_0(1))^2}{G''(0)v_0^3(1)(v_0^2(z) - v_0^2(1))} \end{aligned}$$

The hypotheses on g near $z = 1$ and the result in Lemma 4.1 ensures $w(z) - w(1)$ is uniformly bounded, and since $g(1)$ can be expressed in terms of $w(1)$ as in the previous lemma, then we obtain a uniform bound for $\|w\|_{L^\infty}$. Moreover, by dividing the above identity by $(1 - z)^{1/2}$ we obtained the desired control of $|||w|||_{1/2}$. ☐

Using all the previous results, we can prove the following theorem on the existence of stationary solutions:

Theorem 4.5 (Existence of stationary solutions for small ε)**.** *There exists a positive constant $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$, the stationary equation*

$$0 = \partial_x(\rho\partial_x(\epsilon\rho - G * \rho)) \tag{4.9}$$

admits a nonzero, bounded and compactly supported solution ρ. Moreover, the diameter of the support of ρ is of order $\epsilon^{1/3}$.

Proof. The equation (4.9) can be reformulated as

$$\epsilon\rho(x) - G * \rho(x) = \text{const} \quad \text{if } x \notin \text{supp}\rho. \tag{4.10}$$

Let

$$\tilde{\rho}(y) := \rho\left(CM + \epsilon^{1/3}y\right) \quad \tilde{F}(y) := \int_{-\inf(\text{supp}\rho)}^{y} \tilde{\rho}(y')dy'.$$

By evaluating (4.10) at $x = CM + \epsilon^{1/3}y$ and by integrating with respect to y we get

$$\tilde{F}(y) = \epsilon^{-1}\int_{\mathbb{R}} H(\epsilon^{1/3}(y-t))\tilde{\rho}(t)dt + \alpha y + \beta, \tag{4.11}$$

for some constants $\alpha, \beta > 0$, where $H' = G$, $H(0) = 0$. The equation (4.11) holds for all y such that $CM + \epsilon^{1/3}y \in \text{supp}\rho$. The change of variable $\tilde{F}(y) = v$ on (4.11) and a suitable choice of α, β as before lead to the equation (4.6) with $\delta = \epsilon^{1/3}$. By regarding this equation as an operator equation, due to the results in Lemmas 4.1, 4.2, 4.3 and 4.4, we can apply the Implicit Function Theorem (cf. [14, Theorem 15.1]) and deduce existence of a solution to (4.6) for small ϵ. ☐

4.3. **Larger diffusion.** We finally turn to the natural question whether we can obtain such stationary solutions also for larger values of ϵ. From a modeling point of view one would expect that for ϵ exceeding a certain threshold value, the repulsive forces (modeled by the diffusion) will dominate the aggregative forces, so that solutions of the evolution problem will decay. As we shall see in the following, this is indeed true, and the critical value for ϵ scales with the L^1-norm of the kernel G, which exhibits the interplay of diffusion and aggregation.

Let us assume that there exists an antisymmetric stationary solution u of (4.3). We now define $v := u - CM$ and integrate the equation exactly as in the previous section, which yields

$$0 = \epsilon(\frac{1}{2} - z) + \int_0^1 \left[H(v(z) - v(\zeta)) - v(z)G(v(1) - v(\zeta))\right] d\zeta.$$

Now we multiply this relation by $\text{sign}(\frac{1}{2} - z)$ and integrate to obtain

$$\begin{aligned} 0 &= \epsilon\int_0^1 |\frac{1}{2} - z|dz + \int_0^1\int_0^1 \left[H(v(z) - v(\zeta)) - v(z)G(v(1) - v(\zeta))\right] d\zeta \ \text{sign}(\frac{1}{2} - z)dz \\ &= \frac{\epsilon}{4} + \int_0^1\int_0^1 H(v(z) - v(\zeta)) \ \text{sign}(\frac{1}{2} - z)d\zeta dz + \int_0^1 |v(z)|dz \int_0^1 G(v(1) - v(\zeta))d\zeta, \end{aligned}$$

where we have used the antisymmetry of v for the last term. Due to the monotonicity of v we know that $H(v(z) - v(\zeta)) \ \text{sign}(\frac{1}{2} - z) \geq 0$ if $\frac{1}{2} \geq z \geq \zeta$ and $\frac{1}{2} \leq z \leq \zeta$. On the remaining part of $[0,1]^2$ we estimate

$$H(v(z) - v(\zeta)) \ \text{sign}(\frac{1}{2} - z) \geq -\frac{1}{2}\sup_{p\in\mathbb{R}} |H(p)| = -\frac{1}{2}\int_0^\infty G(p)dp = -\frac{1}{4}\int_{\mathbb{R}} G(p)dp.$$

Hence with the above identity, we conclude

$$0 \geq \frac{\epsilon}{4} - \frac{3}{16}\int_{\mathbb{R}} G(p)dp,$$

which implies that $\epsilon \leq \frac{3}{4}\int_{\mathbb{R}} G(p)dp$ is a necessary condition for the existence of an antisymmetric stationary solution.

Acknowledgements. The work of MB has been supported by the Austrian National Science Foundation FWF through project SFB F 013 / 08 and the Johann Radon Institute for Computational and Applied Mathematics (RICAM) by the Austrian Academy of Sciences in Linz. A major part of this work has been carried out while MB held a position at the Industrial Mathematics Institute, Johannes Kepler University Linz, and MDF held a position at RICAM. MDF also acknowledges support from the Wittgenstein 2000 Award of Peter A. Markowich and from the Wolfgang Pauli Institute of Vienna. Both authors would like to thank Vincenzo Capasso (University of Milano) for interesting and stimulating discussions. They also acknowledge important suggestions for the improvement of the paper by Luigi Ambrosio (SNS Pisa).

Appendix A. **Minimum principle for a nondegenerate approximation.** In this section we prove a technical result for the equation

$$\partial_t \rho = \partial_x \left(\rho \partial_x [a_\lambda(\rho) - G * \rho + V]\right) \qquad \text{in } \mathbb{R} \times \mathbb{R}^+, \tag{A.1}$$

where

$$a_\lambda(\rho) = \lambda \log \rho + a(\rho), \qquad \lambda > 0,$$

which we introduced in the introduction as a non degenerate approximation to (1.1). We shall require the following assumptions

$$A := \|G''\|_{L^\infty} + \|V''\|_{L^\infty} < +\infty. \tag{A.2}$$

We have the following lemma.

Lemma A.1. *Suppose (A.2) holds and suppose that a satisfies the assumption (SD). Let $\rho(x,t)$ be a solution of (A.1) for $(x,t) \in [-R,R] \times [0,+\infty)$ with zero flux boundary conditions*

$$\partial_x [a_\lambda(\rho) - G * \rho + V] = 0 \qquad x = \pm R$$

having initial datum $\rho_0 \in C^1([-R,R])$ such that $\rho_0(x) \geq \mu > 0$ for all $x \in [-R,R]$. Suppose further that the total mass of ρ_0 is one. Then, $\rho(\cdot,t) \in L^1([-R,R])$ for all $t \geq 0$ and $\rho(x,t) \geq \mu e^{-At}$ for almost all $(x,t) \in [-R,R] \times [0,+\infty)$.

Proof. For fixed $\delta > 0$ let $\eta_\delta : [0,+\infty) \to [0,+\infty)$ be a convex smooth approximation of the negative part function

$$(\rho)^- = \max\{0, -\rho\},$$

i. e. suppose that $\eta_\delta(\rho) \to (\rho)^-$ as $\delta \searrow 0$ for all positive ρ. Due to the regularity of ρ we can estimate

$$\begin{aligned}
&\frac{d}{dt}\int_{-R}^{R} \eta_\delta(\rho(x,t) - \mu e^{-At})dx = \int_{-R}^{R} (\rho_t + A\mu e^{-At})\eta_\delta'(\rho(x,t) - \mu e^{-At})dx \\
&= \int_{-R}^{R} \eta_\delta'(\rho(x,t) - \mu e^{-At})[(\rho(a_\lambda(\rho) - G * \rho + V')_x)_x + A\mu e^{-At}]dx \\
&= -\int_{-R}^{R} \eta_\delta''(\rho(x,t) - \mu e^{-At})\rho a_\lambda'(\rho)\rho_x^2 dx + A\mu e^{-At}\int_{-R}^{R} \eta_\delta'(\rho(x,t) - \mu e^{-At})dx \\
&+ \int_{-R}^{R} [\eta_\delta'(\rho(x,t) - \mu e^{-At})(\rho - \mu e^{-At}) - \eta_\delta(\rho(x,t) - \mu e^{-At})](V'' - G'' * \rho)dx \\
&+ \mu e^{-At}\int_{-R}^{R} (V'' - G'' * \rho)\eta_\delta'(\rho(x,t) - \mu e^{-At})dx.
\end{aligned}$$

Due to the convexity of η_δ the term involving $a_\lambda(\rho)$ above is nonnegative. Moreover, we can use

$$\eta_\delta'(u)u - \eta_\delta(u) \to 0 \quad \text{as} \quad \delta \searrow 0$$

and integrate the above relation with respect to time. We obtain

$$\begin{aligned}&\int_{-R}^{R} (\rho(x,t) - \mu e^{-At})^{-} dx \\ &\quad \le \mu \int_0^t e^{-A\tau} \int_{-R}^{R} \eta_\delta'(\rho(x,\tau) - \mu e^{-A\tau})[A + V'' - G'' * \rho(x,\tau)] dx d\tau.\end{aligned}$$

Due to (A.2) and in view of

$$\|G'' * \rho\|_{L^\infty} \le \|G''\|_{L^\infty} \|\rho\|_{L^1} = \|G''\|_{L^\infty}$$

(which holds because of the conservation of the mass), the right hand side above is nonpositive. This proves that $\rho(t) \ge \mu e^{-At}$ for all $t \ge 0$. □

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Received November 2007; revised May 2008.

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