FULLY PARABOLIC KELLER-SEGEL MODEL FOR CHEMOTAXIS WITH PREVENTION OF OVERCROWDING

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ABSTRACT. In this paper we study a fully parabolic version of the Keller-Segel system in presence of a volume filling effect which prevents blow up of the L^∞ norm. This effect is sometimes referred to as *prevention of overcrowding*. As in the parabolic elliptic version of this model (previously studied in [BDFDS06]), the results in this paper basically infer that the combination of the prevention of overcrowding effect with a linear diffusion for the density of cells implies domination of the diffusion effect for large times. In particular, first we show that both the density of cells and the concentration of the chemical vanish uniformly for large times, then we prove that the density of cells converges in L^1 toward the Gaussian profile of the heat equation as time goes to infinity, with a rate which differs from the rate of convergence to self similarity for the heat equation by an arbitrarily small constant ('quasi sharp rate').

1. INTRODUCTION

In this paper we shall deal with the following parabolic system modeling *chemotaxis with prevention of overcrowding*

$$\begin{cases} \rho_t = \varepsilon\Delta\rho - \nabla(\rho(1-\rho)\nabla S) \\ S_t = \Delta S - S + \rho \end{cases} \tag{1}$$

Here ρ models the density of cells, S is the concentration of the chemical substance (chemoattractant). The parameter $\varepsilon > 0$ models the diffusivity of the cells. The present model is posed on the whole space $\mathbb{R}^N$ with $L^1 \cap L^\infty$ initial data for both ρ and S (plus some further assumption on S, see section 2 for more precise statements). In the sequel we shall present a brief overview of results in the literature concerning with chemotaxis models, by justifying the variants included in (1).
Chemotaxis is the phenomenon by which cells move under the influence of chemical substances in their environment. It has been known and widely studied since first descriptions were done by T.W. Engelmann and W.F. Pfeffer for bacteria in 1881

and 1884, and H.S. Jennings for ciliates in 1906. First mathematical models based on partial differential equations arose from the works of C.S. Patlak in 1953, who derived similar models with applications to the study of long-chain polymers (cf. [Pat53]) and E.F. Keller and L.A. Segel in 1970, who proposed a macroscopic model for aggregation of cellular slime molds (cf. [KS70]). Afterwards, several transport phenomena in biological systems have been labeled with the term *chemotaxis*, such as the bacteria Escherichia Coli, or the amoebae Dyctiostelium Discoideum, or endothelial cells of the human body responding to angiogenic factors secreted by a tumor. The main feature of these systems (in a very simplified form involving only two species) is the motion of a species ρ being biased by linear diffusion modeling random motion, with a diffusivity $\varepsilon > 0$ and by the gradient of a certain chemical substance S, whereas the flow of S features birth/death mechanism without cross–diffusion. More precisely, one usually deals with solutions to the Cauchy problem on $\mathbb{R}^N$ for the system

$$\begin{cases} \rho_t = \varepsilon\Delta\rho - \mathrm{div}\,(\rho\chi(\rho, S)\nabla S) \\ S_t = \Delta S + r(\rho, S). \end{cases} \tag{2}$$

In system (2), the birth/death mechanisms for S are contained in the term $r(\rho, S)$. A typical form is the linear one $r(\rho, S) = \alpha\rho - \beta S$ with $\alpha, \beta > 0$. The term $\chi(\rho, S)$, called *chemotactic sensitivity*, is very important in this context. In many situations it turns out that the expression of $\chi(\rho, S)$ determines the final outcome of the competition between diffusion (repulsion of particles) and singular aggregation phenomena (concentration to deltas) at the level of ρ (cf. the works of Jäger–Luckaus [JL92], Nagai [Nag95], Herrero–Velazquez [HV96] among others). In the case $\chi(\rho, S) \equiv$ constant, the above system has been extensively studied, especially in its parabolic–elliptic variants with the second equation in (2) replaced by $0 = \Delta S + \rho - S$ or by Poisson's equation $-\Delta S = \rho$. In particular, it is well known (cf. [DP04]) that the 2 dimensional Keller Segel system

$$\begin{cases} \rho_t = \Delta\rho - \mathrm{div}\,(\rho\chi\nabla S) \\ 0 = \Delta S + \rho \end{cases} \tag{3}$$

(with L^1_+ data for ρ) features a χ–dependent critical threshold m^* for the total mass of ρ determining finite time blow–up or global existence (blow–up for initial mass larger than m^*, global existence otherwise). Related results are contained in [CPZ04, Per04, BDP06] and in the recent preprint [CC08] for the parabolic case.
How to avoid finite time blow–up of cells has been the aim of an extensive research in the last years. This issue is motivated both by the attempt of constructing of an 'approximate' notion of solution preventing blow up for any initial mass on the one side, and by modeling issues related with *volume filling effects* occurring when the density of cells becomes very large on the other side. There are mainly two ways to prevent blow up of ρ. The first one introduces a volume filling effect at the level of the diffusion of cells, replacing $\Delta\rho$ by a nonlinear diffusion term $\Delta\rho^\gamma$ with $\gamma > 1$. This modification of the model (cf. [Kow05, CC06]) allows to define a global solution $\rho(t) \in L^1 \cap L^\infty$ for all $t > 0$ no matter how large the initial mass is. The second way to prevent blow up consists in modifying the chemotactical sensitivity. Among the possible ways to do that (cf. [HPS07, BDP06]), we mention the one suggested by Hillen and Painter in [PH03], which considers $\chi(\rho) = \rho_{max} - \rho$ for a certain $\rho_{max} > 0$ representing the maximum allowed density (ρ_{max} can be

taken equal to 1 for simplicity). Basically, in this model cells stop aggregating when the density reached a maximum allowed value. An extensive mathematical theory for this model with prevention of overcrowding has been performed first in [DS05] on bounded domains and then in [BDFDS06], where also a variant with nonlinear diffusion has been considered in order to stop any mobility mechanism (including diffusion) at a certain density. Both [DS05] and [BDFDS06] concern with the parabolic elliptic model. In this paper we try to generalize some of the results in [BDFDS06] to the fully parabolic model (1). In particular, we aim to prove large time decay of solutions and the large time self–similar behavior of the density of cells. This last issue is extremely non trivial, because of the strong coupling between the two species. In order to perform this task, we use a diffusive time dependent scaling and a variant of the relative entropy method going back to [AMTU00, CJM+01].
The paper is organized as follows: section 2 is devoted to the existence theory. First we prove the existence and uniqueness of solution locally in time for any initial condition, and then we provide some a priori estimates which we will use to prove the existence of a global solution for (1). In section 3 we concern about the long time behavior of the solutions and establish decay rates in $\mathrm{L}^2(\mathbb{R}^n)$ and $\mathrm{L}^\infty(\mathbb{R}^n)$ for both the cells density and the chemical. Finally, in section 4 we study the asymptotic self–similar behavior of the solutions by time dependent scaling and by proving convergence to a stationary state in the new variables.

2. Existence and Regularity

Our aim in this section is to prove the existence and uniqueness of solutions for the Cauchy problem for the parabolic system (1). We will use a fix point argument to show that a unique solution exists locally in time and then we shall provide some estimates to extend this solution globally in time. For future use, we introduce the functional space

$$\mathcal{U} := (\mathrm{L}^1(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)) \times (\mathrm{W}^{1,1}(\mathbb{R}^N) \cap \mathrm{W}^{1,\infty}(\mathbb{R}^N)). \tag{4}$$

As a first step, for a given $(\rho_0, S_0) \in \mathcal{U}$ we will rewrite the system (1) in its integral form

$$\rho(x,t) = \mathcal{G}(x,t) * \rho_0 - \int_0^t \mathcal{G}(t-\tau) * \nabla(\rho(1-\rho)\nabla S)(\tau)\mathrm{d}\tau \tag{5}$$

$$S(x,t) = \mathrm{e}^{-t}\mathcal{G}(x,t) * S_0(x) + \int_0^t \mathrm{e}^{-t+\tau}\mathcal{G}(x,t-\tau) * \rho(x,\tau)\mathrm{d}\tau, \tag{6}$$

where

$$\mathcal{G}(x,t) = \frac{1}{(4\pi t)^{N/2}}\mathrm{e}^{-\frac{|x|^2}{4t}},$$

which is easily obtained by using Duhamel's Formula. These equations leads to the definition of a functional on $\mathcal{U}$ as follows

$$\mathcal{T}[\rho, S] = (\mathcal{T}_1[\rho, S], \mathcal{T}_2[\rho, S]),$$

with

$$\mathcal{T}_1[\rho,S](t) = \mathcal{G}(t) * \rho_0 - \int_0^t \mathcal{G}(t-\tau) * \nabla(\rho(1-\rho)\nabla S)(\tau)\mathrm{d}\tau \tag{7}$$

$$\mathcal{T}_2[\rho,S](t) = \mathrm{e}^{-t}\mathcal{G}(t) * S_0 + \int_0^t \mathrm{e}^{t-\tau}\mathcal{G}(t-\tau) * \rho(\tau)\mathrm{d}\tau. \tag{8}$$

Let us introduce the notation

$$X_T^R = \{(\rho,S) \in \mathcal{U} \ : \ \|(\rho,S)\|_{X_T} \leq R\} \tag{9}$$

where

$$\begin{aligned}\|(\rho,S)\|_{X_T} := \sup_{0\leq t\leq T}\Big\{ & \|\rho(t) - \mathcal{G}(t)*\rho_0\|_1 + \|\rho(t) - \mathcal{G}(t)*\rho_0\|_\infty \\ & + \|S(t) - \mathrm{e}^{-t}\mathcal{G}(t)*S_0\|_1 + \|S(t) - \mathrm{e}^{-t}\mathcal{G}(t)*S_0\|_\infty \\ & + \|\nabla S - \nabla(\mathrm{e}^{-t}\mathcal{G}(t)*S_0)\|_1 + \|\nabla S - \nabla(\mathrm{e}^{-t}\mathcal{G}(t)*S_0)\|_\infty \Big\}\end{aligned}$$

We will prove that X_T^R is invariant under the map $\mathcal{T}$ for T sufficiently small. Then we show that $\mathcal{T}$ is a strict contraction on X_T^R, whence we have the following theorem.

Theorem 2.1. *Let $(\rho_0, S_0) \in \mathcal{U}$. Then, there exists $T > 0$ and a pair $(\rho, S) \in \mathcal{C}([0,T];\mathcal{U})$ such that (ρ,S) solves (5)-(6) in X_T^R and it is unique.*

Proof.- In order to prove the invariance of X_T^R we shall provide a suitable bound for each of the quantities we have to take into account to compute $\|\mathcal{T}\|_{X_T}$. Let $(\rho,S) \in X_T^R$. For the sake of completeness, we shall compute the first bound in detail.

$$\begin{aligned}\|(\mathcal{T}_1[\rho,S](t) - \mathcal{G}*\rho_0)(t)\|_1 &\leq \int_0^t \|\nabla\mathcal{G}(t-\tau)\|_1\|(\rho(1-\rho)\nabla S)(\tau)\|_1\mathrm{d}\tau \\ &\leq \int_0^t C(t-\tau)^{-\frac{1}{2}}\|\nabla S(\tau)\|_1(\|\rho(\tau)\|_\infty + \|\rho(\tau)\|_\infty^2)\mathrm{d}\tau \\ &\leq \int_0^t C(t-\tau)^{-\frac{1}{2}}(\|\nabla(\mathrm{e}^{-\tau}\mathcal{G}(\tau)*S_0)\|_1 + R)\times \\ &\qquad (\|\mathcal{G}(\tau)*\rho_0\|_\infty + R)(\|\mathcal{G}(\tau)*\rho_0\|_\infty + R + 1)\mathrm{d}\tau \\ &\leq C(R, \|\rho_0\|_\infty, \|\nabla S_0\|_1)t^{\frac{1}{2}}\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}\|(\mathcal{T}_1[\rho,S](t) - \mathcal{G}*\rho_0)(t)\|_\infty &\leq C(R, \|\rho_0\|_\infty, \|\nabla S_0\|_\infty)t^{\frac{1}{2}} \\ \|(\mathcal{T}_2[\rho,S](t) - \mathrm{e}^{-t}\mathcal{G}*S_0)(t)\|_1 &\leq C(R, \|\rho\|_1)(1-e^{-t}) \\ \|(\mathcal{T}_2[\rho,S](t) - \mathrm{e}^{-t}\mathcal{G}*S_0)(t)\|_\infty &\leq C(R, \|\rho\|_\infty)(1-e^{-t})\end{aligned}$$

Finally,

$$\begin{aligned}\|\nabla[\mathcal{T}_2(t) - \mathrm{e}^{-t}\mathcal{G}(t)*S_0]\|_1 &\leq \int_0^t \mathrm{e}^{-t+\tau}\|\nabla\mathcal{G}(t-\tau)*\rho(\tau)\|_1 \\ &\leq \int_0^t \mathrm{e}^{-t+\tau}(t-\tau)^{-\frac{1}{2}}\|\rho(\tau)\|_1\mathrm{d}\tau \leq C(R, \|\rho_0\|_1)t^{\frac{1}{2}},\end{aligned}$$

and in the same spirit,

$$\|\nabla[\mathcal{T}_2 - \mathrm{e}^{-t}\mathcal{G} * S_0]\|_\infty \leq C(R, \|\rho_0\|_\infty) t^{\frac{1}{2}}.$$

Therefore, for a given $(\rho, S) \in X_T^R$ we have that

$$\|\mathcal{T}[\rho, s]\|_{X_T} \leq C(R, \|\rho_0\|_1, \|\rho_0\|_\infty, \|\nabla S_0\|_1, \|\nabla S_0\|_\infty) T^{\frac{1}{2}},$$

whence the invariance of X_T^R under $\mathcal{T}$ if T is small enough. Now we want to see that $\mathcal{T}$ is strictly contractive on X_T^R. For that let us consider two pairs (ρ_1, S_1) and (ρ_2, S_2) belonging to X_T^R and look at the norm of the difference of their images by $\mathcal{T}$ in $\mathcal{C}([0,T];\mathcal{U})$.

$$\begin{aligned}
&\|(\mathcal{T}_1[\rho_1, S_1] - \mathcal{T}_1[\rho_2, S_2])(t)\|_1 \leq \\
&\qquad \leq \int_0^t \|\nabla \mathcal{G}(t-\tau) * [\rho_1(1-\rho_1)\nabla S_1 - \rho_2(1-\rho_2)\nabla S_2]\,(\tau)\|_1 \mathrm{d}\tau \\
&\qquad \leq \int_0^t (t-\tau)^{-\frac{1}{2}} \|\,[\rho_1(1-\rho_1)\nabla S_1 - \rho_2(1-\rho_2)\nabla S_2 \\
&\qquad\quad + \rho_2 \nabla S_1(\rho_2 + (1-\rho_1)) - \rho_2 \nabla S_1(\rho_2 + (1-\rho_1))]\,(\tau)\|_1 \mathrm{d}\tau
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^t (t-\tau)^{-\frac{1}{2}} \Big[\|(1-\rho_1)\nabla S_1\|_\infty \|\rho_1 - \rho_2\|_1 + \|\rho_2\|_\infty \|\nabla S_1 - \nabla S_2\|_1 \\
&\qquad + \|\rho_2^2\|_\infty \|\nabla S_1 - \nabla S_2\|_1 + \|\rho_2 \nabla S_1\|_\infty \|\rho_1 - \rho_2\|_1\Big] \\
&\leq C(R, \|\rho_1\|_\infty, \|\rho_2\|_\infty, \|\nabla S_1\|_\infty, \|\nabla S_2\|_\infty) T^{\frac{1}{2}} \left(\sup_{0 \leq t \leq T} \|(\rho_1 - \rho_2)(t)\|_1 \right. \\
&\qquad \left. + \sup_{0 \leq t \leq T} \|(\nabla S_1 - \nabla S_2)(t)\|_1 \right)(\tau) \mathrm{d}\tau
\end{aligned}$$

In the same way we see that

$$\begin{aligned}
\|(\mathcal{T}_1[\rho_1, S_1] - \mathcal{T}_1[\rho_2, S_2])(t)\|_\infty \quad &\leq \quad C(R, \|\rho_1\|_\infty, \|\rho_2\|_\infty, \|\nabla S_1\|_\infty, \|\nabla S_2\|_\infty) T^{\frac{1}{2}} \times \\
&\qquad \left(\sup_{0 \leq t \leq T} \|(\rho_1 - \rho_2)(t)\|_\infty + \sup_{0 \leq t \leq T} \|(\nabla S_1 - \nabla S_2)(t)\|_\infty \right)
\end{aligned}$$

and

$$\begin{aligned}
\|(\mathcal{T}_2[\rho_1, S_1] - \mathcal{T}_2[\rho_2, S_2])(t)\|_1 \quad &\leq \quad C(1 - \mathrm{e}^{-T}) \sup_{0<t<T} \|(\rho_1 - \rho_2)(t)\|_1 \\
\|(\mathcal{T}_2[\rho_1, S_1] - \mathcal{T}_2[\rho_2, S_2])(t)\|_\infty \quad &\leq \quad C(1 - \mathrm{e}^{-T}) \sup_{0<t<T} \|(\rho_1 - \rho_2)(t)\|_\infty. \\
\|\nabla(\mathcal{T}_2[\rho_1, S_1] - \mathcal{T}_2[\rho_2, S_2])(t)\|_1 \quad &\leq \quad C T^{\frac{1}{2}} \sup_{0<t<T} \|(\rho_1 - \rho_2)(t)\|_1 \\
\|\nabla(\mathcal{T}_2[\rho_1, S_1] - \mathcal{T}_2[\rho_2, S_2])(t)\|_\infty \quad &\leq \quad C T^{\frac{1}{2}} \sup_{0<t<T} \|(\rho_1 - \rho_2)(t)\|_\infty.
\end{aligned}$$

Hence, if T is small we have $\|\mathcal{T}[\rho_1, S_1] - \mathcal{T}[\rho_2, S_2]\|_{\mathcal{U}} \leq \alpha \|(\rho_1, S_1) - (\rho_2, S_2)\|_{\mathcal{U}}$ for $0 < \alpha < 1$ which implies the contractivity of $\mathcal{T}$ and concludes the proof. $\square$

At this point we shall remark some properties about the solution to (1), namely we notice that the mass of ρ is preserved and that the interval $[0,1]$ is an invariant domain for ρ. We collect these properties in the next proposition.

Proposition 2.2. *Let $\rho_0 \in \mathrm{L}^1(\mathbb{R}^N)$ such that $0 \le \rho_0 \le 1$; then for any t, $0 \le t \le T$*

$$\int_{\mathbb{R}^N} \rho(x,t)\mathrm{d}x = \int_{\mathbb{R}^N} \rho_0(x,t)\mathrm{d}x.$$

and $0 \le \rho(t) \le 1$ for $0 \le t \le T$.

Proof.- Concerning the first part of the proposition, we consider a family of non-increasing cut-off functions $\{\zeta_n\}$; then multiply (1) by ζ_n and integrate over $\mathbb{R}^N \times [0,T]$. The result follows from dominated convergence theorem when we let n go to ∞. For the second part, it is easy to see that $\rho \equiv 0$ and $\rho \equiv 1$ are sub- and super-solutions respectively, so if initially ρ belongs to the interval $[0,1]$ it will remain there (see [PW84]). ◻

Then we are ready to state the existence of a global and unique solution for (1)

Theorem 2.3. *Let $(\rho_0, S_0) \in \mathcal{U}$, $0 < \rho_0 < 1$. Then there exist a unique weak solution for (1) defined in $[0,\infty)$ which belongs to $\mathcal{U}$ for each $T > 0$.*

Proof.- From theorem 2.1 we know that there exist a unique weak solution for (1) defined in $(0,T)$ for some $T > 0$. By contradiction, let $T_{\max} > 0$ be the maximal time of existence of (ρ, S). Then an easy continuation argument shows that $\|(\rho,S)\|_{X_t}$ should go to ∞ as $t \to T_{\max}$. But the conservation of the mass and the existence of the sub and super–solutions for ρ tell us that ρ is uniformly bounded in $\mathrm{L}^1(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$ and then global in time. $\mathrm{L}^1 \cap \mathrm{L}^\infty$ bounds for S follows easily. Also, using the L^∞ estimates for ρ in the integral expression (6), one can easily show that for any finite time both the L^1 and the L^∞ norm of ∇S are bounded. Therefore, such $T_{\max}$ cannot exist and the thesis follows. ◻

3. Decay Rates for the concentration of cells and the chemical

Here we want to provide a decay rate for the L^∞-norm of the density of cells ρ. Also a bound for the L^∞-norm of the gradient of the concentration of the chemical will be derived. Our first goal will be to find a decay rate for the L^2-norm and the L^∞-norm of ρ so that we can give a bound for the decay of $\|\rho\|_p$ by interpolation. Next proposition provides the decay of L^2-norm of ρ, and with the same effort, we will obtain too an estimate for the decay of the L^2-norm of ∇S. From now on, we shall need the assumption on the diffusivity constant $\varepsilon > \frac{1}{4}$. Without this assumption we are not able to prove the same decay estimates. We remark that the same restriction is present in [DS05, BDFDS06]. Moreover, even in case $\varepsilon < \frac{1}{4}$ this model does not feature nontrivial steady states, therefore the question of the long time behavior for $\varepsilon < \frac{1}{4}$ is still open, although numerical simulations in [BDFDS06] still suggest large time decay.

Proposition 3.1. *Let $\varepsilon > \frac{1}{4}$. Let (ρ, S) be a solution of the parabolic problem (1) with initial datum (ρ_0, S_0) satisfying $\rho_0, \nabla S_0 \in L^2(\mathbb{R}^N)$, then there exists $\lambda > 0$ such that*

$$\|\rho(t)\|_2 + \lambda\|\nabla S(t)\|_2 \le C(t+1)^{-\frac{N}{4}} \tag{10}$$

Proof.-
To get this result we look at the time evolution of $\mathcal{E}[\rho,S]:=\frac{1}{2}\left(\|\rho\|_2^2+\lambda^2\|\nabla S\|_2^2\right)$. Recalling (1) we can see that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N}\left[\frac{\rho^2}{2}+\lambda\frac{|\nabla S|^2}{2}\right] &= \int_{\mathbb{R}^N}\rho_t\rho\ \mathrm{d}x+\lambda\int_{\mathbb{R}^N}\nabla S\nabla S_t\ \mathrm{d}x\\ &= \int_{\mathbb{R}^N}\rho(\varepsilon\Delta\rho-\nabla(\rho(1-\rho)\nabla S))\ \mathrm{d}x-\lambda\int_{\mathbb{R}^N}\Delta S(\Delta S-S+\rho)\ \mathrm{d}x\\ &= -\varepsilon\int_{\mathbb{R}^N}|\nabla\rho|^2\mathrm{d}x+\int_{\mathbb{R}^N}\rho(1-\rho)\nabla S\nabla\rho\mathrm{d}x-\lambda\int_{\mathbb{R}^N}(\Delta S)^2\mathrm{d}x\\ &\quad -\lambda\int_{\mathbb{R}^N}|\nabla S|^2\mathrm{d}x+\lambda\int_{\mathbb{R}^N}\nabla S\nabla\rho\mathrm{d}x\\ &\le -\varepsilon\int_{\mathbb{R}^N}|\nabla\rho|^2\mathrm{d}x+\left(\frac{1}{4}+\lambda\right)\int_{\mathbb{R}^N}|\nabla S||\nabla\rho|\mathrm{d}x-\lambda\int_{\mathbb{R}^N}|\nabla S|^2\mathrm{d}x\\ &= -a\int_{\mathbb{R}^N}|\nabla\rho|^2\mathrm{d}x-\int_{\mathbb{R}^N}\left(b|\nabla\rho|-c|\nabla S|\right)^2\mathrm{d}x \end{aligned}\tag{11}$$

where we have set

$$c=\sqrt{\lambda}\qquad b=\frac{\frac{1}{4}+\lambda}{2\sqrt{\lambda}}\qquad a=\varepsilon-b^2=\varepsilon-\left(\frac{\frac{1}{4}+\lambda}{2\sqrt{\lambda}}\right)^2\tag{12}$$

For $\varepsilon>\frac{1}{4}$ there exists an interval $\mathcal{I}_\varepsilon$ such that for $\lambda\in\mathcal{I}_\varepsilon$, a is positive, and thus both terms in the right hand side of (11) are negative, whence the L^2-norm of ρ and ∇S is not increasing. We want to see that in fact they are decaying with a suitable rate. In order to prove that, we notice that from (11) the following inequality follows

$$\frac{d}{dt}\int_{\mathbb{R}^N}\left[\frac{\rho^2}{2}+\lambda\frac{|\nabla S|^2}{2}\right]\le -a\int_{\mathbb{R}^N}|\nabla\rho|^2\mathrm{d}x-\frac{c}{2}\int_{\mathbb{R}^N}|\nabla S|^2\mathrm{d}x,\tag{13}$$

which can be written as

$$\mathcal{E}(\rho(t),S(t))+\int_0^t\left[a\int_{\mathbb{R}^N}|\nabla\rho|^2\mathrm{d}x+\frac{c}{2}\int_{\mathbb{R}^N}|\nabla S|^2\mathrm{d}x\right]\mathrm{d}\tau\le\mathcal{E}(\rho_0,S_0).\tag{14}$$

Then, in one hand, by the following interpolation inequality (see [EZ91])

$$\|\rho\|_{\mathrm{L}^p(\mathbb{R}^N)}^{\frac{(N(p-1)+2)p}{N(p-1)}}\le C(p,N)\|\nabla\rho^{\frac{p}{2}}\|_{\mathrm{L}^p(\mathbb{R}^N)}^2\|\rho\|_{\mathrm{L}^1(\mathbb{R}^N)}^{\frac{2p}{d(p-1)}}\tag{15}$$

we have that

$$-a\int_{\mathbb{R}^N}|\nabla\rho|^2\mathrm{d}x\le -a\frac{\|\rho\|_{\mathrm{L}^2(\mathbb{R}^N)}^{\frac{2(N+2)}{N}}}{C(p,N)\|\rho\|_{\mathrm{L}^1(\mathbb{R}^N)}^{\frac{4}{N}}}.\tag{16}$$

Now, since (14) is valid for all $t\ge 0$, we have

$$\int_0^\infty\left[a\int_{\mathbb{R}^N}|\nabla\rho|^2\mathrm{d}x+\frac{c}{2}\int_{\mathbb{R}^N}|\nabla S|^2\mathrm{d}x\right]\mathrm{d}\tau<+\infty$$

Then, there exists a sequence $t_k\to+\infty$ such that

$$\left[a\int_{\mathbb{R}^N}|\nabla\rho|^2\mathrm{d}x+\frac{c}{2}\int_{\mathbb{R}^N}|\nabla S|^2\mathrm{d}x\right](t_k)\to 0$$

as $k \to +\infty$, which implies

$$\int_{\mathbb{R}^N} |\nabla \rho(x, t_k)|^2 \mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla S(x, t_k)|^2 \mathrm{d}x \to 0$$

and therefore, using again inequality (15),

$$\int_{\mathbb{R}^N} \rho^2(x, t_k) \mathrm{d}x \to 0.$$

Hence, $\mathcal{E}(\rho(t_k), S(t_k)) \to 0$ as $k \to \infty$, but since $\mathcal{E}$ is non-increasing w.r.t time, we get that in fact $\mathcal{E}(\rho(t), S(t)) \to 0$ as $t \to \infty$. In particular, this implies that $\int_{\mathbb{R}^N} |\nabla S(x,t)|^2 \mathrm{d}x \to 0$ as $t \to \infty$ and for big enough t

$$-\frac{c}{2} \int_{\mathbb{R}^N} |\nabla S|^2 \mathrm{d}x \leq -\frac{c}{2} \left(\int_{\mathbb{R}^N} |\nabla S|^2 \mathrm{d}x \right)^{\alpha},$$

for $\alpha > 1$. Thus from (13) we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} \left[\frac{\rho^2}{2} + \lambda \frac{\nabla S^2}{2} \right] \mathrm{d}x &\leq -C \left[\left(\int_{\mathbb{R}^N} \rho^2 \mathrm{d}x \right)^{\frac{(N+2)}{N}} + \left(\int_{\mathbb{R}^N} |\nabla S|^2 \mathrm{d}x \right)^{\frac{(N+2)}{N}} \right]. \\ &\leq -C \left(\int_{\mathbb{R}^N} \rho^2 \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla S|^2 \mathrm{d}x \right)^{\frac{(N+2)}{N}} \end{aligned} \tag{17}$$

Now, by time integration of the previous expression the thesis follows. □

At this point, we only need to prove an estimate for the decay of the L^∞ norm of ρ. From now on, due to the integrability problems that higher dimension entails, we restrict ourselves to the one-dimensional case. We look at the expression of ρ for a time $2t$ in terms of its value at time t given by Duhamel's formula

$$\rho(x, 2t) = \mathcal{G}(t) * \rho(t) + \int_0^t \nabla \mathcal{G}(t-s) * ((\rho(1-\rho))\nabla S)(t+\tau) d\tau \tag{18}$$

so we can estimate

$$\begin{aligned} \|\rho(2t)\|_\infty &\leq \|\mathcal{G}(t) * \rho(t)\|_\infty + \int_0^t \|\mathcal{G}_x(t-\tau) * ((\rho(1-\rho))S_x)(t+\tau)\|_\infty \mathrm{d}\tau \\ &\leq \|\mathcal{G}(t)\|_\infty \|\rho(t)\|_1 + \int_0^t \|\nabla \mathcal{G}(t-\tau)\|_2 \|\rho S_x\|_2 \mathrm{d}\tau \\ &\leq C(t-\tau)^{-\frac{1}{2}} \|\rho\|_1 + \int_0^t C(t-\tau)^{-\frac{3}{4}} \|\rho\|_4 \|S_x\|_4 \mathrm{d}\tau \end{aligned} \tag{19}$$

Thus, to get the decay we need to compute an estimate for $\|S_x\|_4$ and $\|\rho\|_4$. Let us start by $\|\rho\|_4$. Similarly as before we can estimate :

$$\begin{aligned}
\|\rho(2t)\|_4 &\le \|\mathcal{G}(t)*\rho(t)\|_4 + \int_0^t \|\mathcal{G}_x(t-\tau)*((\rho(1-\rho))S_x)(t+\tau)\|_4 \mathrm{d}\tau \\
&\le \|\mathcal{G}(t)\|_4\|\rho(t)\|_1 + \int_0^t \|\mathcal{G}_x(t-\tau)\|_4\|\rho S_x\|_1 \mathrm{d}\tau \\
&\le Ct^{-\frac{3}{8}}\|\rho\|_1 + \int_0^t C(t-\tau)^{-\frac{7}{8}}\|\rho\|_2\|S_x\|_2 \mathrm{d}\tau \\
&\le Ct^{-\frac{3}{8}}\|\rho\|_1 + \int_0^t \hat{C}(t-\tau)^{-\frac{7}{8}}(t+\tau)^{-\frac{1}{2}} \mathrm{d}\tau \\
&\le Ct^{-\frac{3}{8}}\|\rho\|_1 + \hat{C}t^{-\frac{3}{8}} = C(M)t^{-\frac{3}{8}} \qquad (20)
\end{aligned}$$

($C(M)$ is a constant depending on the total mass M of ρ) and due to the integral equation (6) satisfied by S, we have

$$\begin{aligned}
\|S_x\|_4 &\le \mathrm{e}^{-t}\|\mathcal{G}_x(t)\|_4\|(S_0)_x\|_1 + \int_0^t \mathrm{e}^{-t+\tau}\|\mathcal{G}_x(t-\tau)\|_1\|\rho(\tau)\|_4 \mathrm{d}\tau \\
&\le Ct^{-\frac{3}{8}}\|(S_0)_x\|_1 + \int_0^t \mathrm{e}^{\tau-t}(t-\tau)^{-\frac{1}{2}}(t+\tau)^{-\frac{3}{8}} \mathrm{d}\tau \\
&\le t^{-\frac{3}{8}}\left(C\|(S_0)_x\|_1 + \int_0^t \mathrm{e}^{\tau-t}(t-\tau)^{-\frac{1}{2}}\right)\mathrm{d}\tau \\
&= C(\|(S_0)_x\|_1)t^{-\frac{3}{8}}. \qquad (21)
\end{aligned}$$

Now we can continue from (19) and finish the computation:

$$\begin{aligned}
\|\rho(2t)\|_\infty &\le Ct^{-\frac{1}{2}}\|\rho\|_1 + \int_0^t C(t-\tau)^{-\frac{3}{4}}C(M,\|(S_0)_x\|_1)(t+\tau)^{-\frac{6}{8}} \mathrm{d}\tau \\
&\le Ct^{-\frac{1}{2}}\|\rho\|_1 + \int_0^t C(t-\tau)^{-\frac{3}{4}}C(M,\|(S_0)_x\|_1)(t)^{-\frac{6}{8}} \mathrm{d}\tau \\
&\le Ct^{-\frac{1}{2}}\|\rho\|_1 + \widetilde{C}t^{-\frac{6}{8}+\frac{1}{4}} = Ct^{-\frac{1}{2}}. \qquad (22)
\end{aligned}$$

and with the same idea we can also see that S is decaying

$$S(2t) = e^{-t}\mathcal{G}(t)*S(t) + \int_0^t e^{-t+s}\mathcal{G}(x,t-s)*\rho(x,t+\tau)\mathrm{d}\tau \qquad (23)$$

so

$$\begin{aligned}
\|S(2t)\|_\infty &\le C\mathrm{e}^{-t}\|S(t)\|_\infty + C\int_o^t \mathrm{e}^{-t+\tau}(t+\tau)^{-\frac{1}{2}} \\
&\le C\mathrm{e}^{-t}\|S(t)\|_\infty + Ct^{-\frac{1}{2}}(1-\mathrm{e}^{-t})
\end{aligned}$$

These results can be sumarized in the next

Proposition 3.2. *Let $\varepsilon > \frac{1}{4}$ and $N = 1$. Let the pair (ρ, S) be solution of (1) with initial datum $(\rho_0, S_0) \in U$ such that $0 \le \rho_0 \le 1$. Then $\|\rho\|_\infty = \mathrm{O}(t^{-\frac{1}{2}})$ and $\|S\|_\infty = \mathrm{O}(t^{-\frac{1}{2}})$ as $t \to +\infty$.*

Remark 3.3. It is clear from last estimate above that the L^∞ assumptions on $(S_0)_x$ could be slightly relaxed. We shall not deal with this issue for the sake of simplicity.

4. Asymptotic self-similar behavior

Once we know that there exists a time–decaying solution for the fully parabolic problem (1) from previous section, in this one we will be concerned about its long-time asymptotics. For simplicity, we will assume ε to be equal to 1, and show by means of a time dependent scaling and entropy dissipation tools that as time grows to infinity the solution of (1) converges in L^1 towards the following time translated self-similar gaussian solution of the Heat equation

$$\rho^\infty(t) = \frac{C_M}{(4\pi(2t+1))^{1/2}} e^{-\frac{|x|^2}{2(2t+1)}}. \tag{24}$$

For that let us consider the scaling

$$\left\{ \begin{array}{rcl} \rho(x,t) & = & (2t+1)^{-\frac{1}{2}} v(y,\tau) \\ S(x,t) & = & (2t+1)^{-\frac{1}{2}} \sigma(y,\tau) \\ y(x,t) & = & x(2t+1)^{-\frac{1}{2}} \\ \tau(x,t) & = & \frac{1}{2}\log(2t+1) \end{array} \right. \tag{25}$$

so that (1) becomes

$$\left\{ \begin{array}{rcl} v_\tau & = & (yv)_y + v_{yy} - \mathrm{e}^{-\tau}[v(1-\mathrm{e}^{-\tau}v)\sigma_y]_y \\ \sigma_\tau & = & (y\sigma)_y + \sigma_{yy} + \mathrm{e}^{2\tau}(v-\sigma) \end{array} \right. . \tag{26}$$

Also, we define the entropy functional for the v variable

$$E(v) = \int_{\mathbb{R}} v \left(\log v + \frac{y^2}{2} \right) \mathrm{d}y. \tag{27}$$

This functional admits a unique global minimum v_M^∞ in the space of L^1_+ densities with prescribed mass M. More precisely,

$$v^\infty = C_M \mathrm{e}^{-\frac{y}{2}} \tag{28}$$

is the scaled gaussian and the constant C_M depends on the total mass M of v. With these settings we are ready to prove the following theorem.

Theorem 4.1. *Let $N = \varepsilon = 1$ and let (ρ, S) be the solution to (1) with initial condition (ρ_0, S_0) satisfying the assumptions of theorem 2.3 and let $\rho^\infty(t)$ be defined by (24). Let (v,σ) be defined by (26) and v^∞ as in (28). Then, for any arbitrarily small $\delta > 0$ there exist a constant C depending on δ and on the initial data such that*

$$\|v(\tau) - v^\infty\|_1 \le C\mathrm{e}^{-(1-\delta)\tau} \tag{29}$$

for all $\tau > 0$, or equivalently

$$\|\rho(t) - \rho^\infty(t)\|_1 \le C(t+1)^{-\frac{1-\delta}{2}} \tag{30}$$

for all $t > 0$.

Proof.- First, let us introduce the short notation

$$W = \left(\log v + \frac{y^2}{2} \right)_y = \frac{v_y}{v} + y \tag{31}$$

by which we can write the scaled problem as

$$\left\{ \begin{array}{rcl} v_\tau & = & (vW)_y - \mathrm{e}^{-\tau}(v(1-\mathrm{e}^{-\tau}v)\sigma_y)_y \\ \sigma_\tau & = & \sigma_{yy} + (y\sigma)_y + \mathrm{e}^{2\tau}(v-\sigma) \end{array} \right. \tag{32}$$

The proof is based on the two following lemmas, which provide us with the dissipation of the entropy functional (27) and $\|\sigma_y\|_2^2$.

Lemma 4.2. *For all $\delta \in (0,1)$ we have*

$$\frac{d}{d\tau}E(v) \leq -(1-\delta)\int_{\mathbb{R}} vW^2 \mathrm{d}y + \frac{\mathrm{e}^{-2\tau}}{4\delta}\int_{\mathbb{R}} v\sigma_y^2 \mathrm{d}y \tag{33}$$

Proof.- We can compute the entropy dissipation by multiplying first equation in (26) by $(\log v + \frac{y^2}{2})$ and integrating by parts to get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} v\left(\log v + \frac{y^2}{2}\right)\mathrm{d}y = \\
&\qquad = -\int_{\mathbb{R}} v\left(\log v + \frac{y^2}{2}\right)_y^2 \mathrm{d}y + \mathrm{e}^{-\tau}\int_{\mathbb{R}} v(1-\mathrm{e}^{-\tau}v)\sigma_y\left(\log v + \frac{y^2}{2}\right)_y \mathrm{d}y \\
&\qquad \leq -(1-\delta)\int_{\mathbb{R}} v\left(\log v + \frac{y^2}{2}\right)_y^2 \mathrm{d}y + \frac{\mathrm{e}^{-2\tau}}{4\delta}\int_{\mathbb{R}} v\sigma_y^2 \mathrm{d}y
\end{aligned} \tag{34}$$

Using the notation introduced in (31) the lemma follows. ∎

Lemma 4.3.

$$\begin{aligned}
\frac{d}{d\tau}\left(\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\right)\mathrm{d}y \leq \mathrm{e}^{-2\tau}\|v\|_\infty\int_{\mathbb{R}} vW^2\mathrm{d}y + 2\mathrm{e}^{-2\tau}\|v\|_\infty\|v\|_1 - \mathrm{e}^{-2\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y \\
- \mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y
\end{aligned} \tag{35}$$

Proof.-

$$\begin{aligned}
\frac{d}{d\tau}\left(\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y\right) &= -4\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y + 2\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y(\sigma_y)_\tau\mathrm{d}y \\
&= -4\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - 2\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_{yy}\left((y\sigma)_y + \sigma_{yy} + \mathrm{e}^{2\tau}(v-\sigma)\right)\mathrm{d}y \\
&= -4\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - 2\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_{yy}^2\mathrm{d}y - 2\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_{yy}(y\sigma)_y\mathrm{d}y \\
&\quad - 2\mathrm{e}^{-2\tau}\int_{\mathbb{R}}\sigma_{yy}(v-\sigma)\mathrm{d}y \\
&= -4\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - 2\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_{yy}^2\mathrm{d}y - 2\mathrm{e}^{-2\tau}\int_{\mathbb{R}}\sigma_{yy}(v-\sigma)\mathrm{d}y + 3\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y \\
&= -\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - 2\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_{yy}^2\mathrm{d}y + 2\mathrm{e}^{-2\tau}\int_{\mathbb{R}}\sigma_y v_y\mathrm{d}y - 2\mathrm{e}^{-2\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y \\
&\leq -\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - 2\mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_{yy}^2\mathrm{d}y + \mathrm{e}^{-2\tau}\int_{\mathbb{R}} v\frac{v_y^2}{v}\mathrm{d}y - \mathrm{e}^{-2\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y \\
&\leq \mathrm{e}^{-2\tau}\|v\|_\infty\int_{\mathbb{R}} vW^2\mathrm{d}y - \mathrm{e}^{-2\tau}\|v\|_\infty\int_{\mathbb{R}} vy^2\mathrm{d}y - 2\mathrm{e}^{-2\tau}\|v\|_\infty\int_{\mathbb{R}} v_y y\mathrm{d}y \\
&\quad - \mathrm{e}^{-2\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - \mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y \\
&\leq \mathrm{e}^{-2\tau}\|v\|_\infty\int_{\mathbb{R}} vW^2\mathrm{d}y + 2\mathrm{e}^{-2\tau}\|v\|_\infty\|v\|_1 - \mathrm{e}^{-2\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - \mathrm{e}^{-4\tau}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y
\end{aligned}$$

□

Now, in view of the uniform decay estimates proven in Proposition 3.2, we are able to find a constant $\mu > 0$ such that $\|v\|_\infty \le \mu$ for all τ. With such μ at hand we introduce the functional

$$\Phi(v,\sigma,\tau) := E(v) - E(v^\infty) + \frac{\mu}{2\delta}\mathrm{e}^{-4\tau}\int \sigma_y^2 \tag{36}$$

Next we compute the evolution of Φ with respect to τ to get

$$\begin{aligned}
\frac{d}{d\tau}\left[E(v) - E(v^\infty) + \frac{\mu}{2\delta}\mathrm{e}^{-4\tau}\int \sigma_y^2\right] \le \\
\le -(1-\delta)\int vW^2 + \frac{\mathrm{e}^{-2\tau}}{4\delta}\int v\sigma_y^2 + \mathrm{e}^{-2\tau}\frac{\mu}{2\delta}\|v\|_\infty\int vW^2 \\
+ 2\mathrm{e}^{-2\tau}\frac{\mu}{2\delta}\|v\|_\infty\|v\|_1 - \mathrm{e}^{-2\tau}\frac{\mu}{2\delta}\int\sigma_y^2 - \mathrm{e}^{-4\tau}\frac{\mu}{2\delta}\int\sigma_y^2 \\
\le -(1-\delta-\mathrm{e}^{-2\tau}\frac{\mu^2}{2\delta})\int vW^2 - \frac{\mathrm{e}^{-2\tau}}{4\delta}\mu\int\sigma_y^2 + 2\mathrm{e}^{-2\tau}\frac{\mu^2}{2\delta}\|v\|_1 \\
- \mathrm{e}^{-4\tau}\frac{\mu}{2\delta}\int\sigma_y^2 \\
\le -(1-\delta-\mathrm{e}^{-2\tau}\frac{\mu^2}{2\delta})\int vW^2 + 2\mathrm{e}^{-2\tau}\frac{\mu^2}{2\delta}\|v\|_1 - \mathrm{e}^{-2\tau}\frac{\mu}{2\delta}\int\sigma_y^2
\end{aligned}$$

which implies

$$\frac{d}{d\tau}\left[E(v) - E(v^\infty) + \frac{\mu}{2\delta}\mathrm{e}^{-4\tau}\int \sigma_y^2\right] \le -(1-2\delta)\int vW^2 + O(\mathrm{e}^{-2\tau}) - A\mathrm{e}^{-4\tau}\frac{\mu}{2\delta}\int\sigma_y^2 \tag{37}$$

for an arbitrarily large A and for $\tau \ge \tau^*$ with τ^* depending on A. Now, notice that due to log-Sobolev inequality (cf. [AMTU00])

$$2(E(v) - E(v^\infty) \le \int vW^2.$$

and thus the previous estimate (37) reads

$$\begin{aligned}
\frac{d}{d\tau}\left[E(v) - E(v^\infty) + \mu\mathrm{e}^{-4\tau}\int\sigma_y^2\right] \\
\le -(2-2\delta)(E(v) - E(v^\infty)) + O(\mathrm{e}^{-2\tau}) - A\mathrm{e}^{-4\tau}\mu\int\sigma_y^2
\end{aligned}$$

for $\tau \ge \tau^*$, i.e., by choosing a properly large A,

$$\frac{d}{d\tau}\Phi(v,\sigma,\tau) \le -(2-2\delta)\Phi(v,\sigma,\tau) + O(\mathrm{e}^{-2\tau}) \tag{38}$$

whence,

$$\Phi(v,\sigma,\tau) \le C\mathrm{e}^{-(2-2\delta)\tau}. \tag{39}$$

Here a suitable constant $C > 0$ (depending on the initial data) can be chosen in such a way that (39) is valid for all $\tau > 0$. This can be done by proving that the modified entropy functional $\Phi(v(\tau),\sigma(\tau),\tau)$ is uniformly bounded on all compact intervals $\tau \in [0, t^*]$.

The statement (29) follows by a Csizar-Kullbak inequality (see [AMTU01]). By going back to the original variables $\rho = \rho(x,t)$ we also recover (30) and the proof is complete. □

Remark 4.4. It is well known that under similar assumptions on the initial data, the heat equation produces a rate of convergence to self similarity in L^1 of the form $t^{-1/2}$ in 1 space dimension. In this sense, we can state that the rate of convergence here is 'quasi sharp'.

Remark 4.5. By comparing our result with the one of ([BDFDS06]) concerning with self similar decay, we recover that the decay rate toward self similarity for the density of cells ρ is the same as in the parabolic elliptic model. Whether S features also a self similar behavior for large times is an open problem, which we shall address in the future.

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