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Abstract

We propose a new class of sharing rules for the distribution of the gains from cooperation for partition function games with externalities. We show that these sharing rules are characterized by three axioms: coalitional efficiency, additivity and anonymity. Moreover, they stabilize, in the sense of d'Aspremont et al. (1983), the coalition which generates the highest global welfare among the set of potentially stable coalitions. Our sharing rules are particularly powerful for economic problems that are characterized by positive externalities from coalition formation (outsiders benefit from the enlargement of coalitions) and which therefore typically suffer from free-riding. Our results also carry over to negative externality games in which cooperation is believed to be easier.
keywords: partition function, coalition formation, externalities, surplus sharing rules
JEL codes: C70, C71

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1 Introduction

It is well-known that the presence of externalities warrants some form of coordination and cooperation between economic agents to prevent welfare losses. By coordinating their strategies, agents can improve, both collectively and individually, upon non-cooperative outcomes provided the gains from cooperation are shared in accordance with the individual interests of all participants. Examples of economic problems with externalities are plentiful in the literature: they range from output and price setting of firms in imperfectly competitive markets (e.g. Deneckere and Davidson 1985 and Yi 1997), R&D investments of firms (e.g. Poyago-Theotokay 1995 and Yi and Shin 2000), the provision of public goods, like greenhouse gas emission reduction, the eradication of contagious diseases, like Malaria and AIDS, and the fight against terrorism (e.g. Sandler 2004 for an overview), to coordination of tariffs and monetary policies (e.g. Yi 1996 and Kohler 2002) among nations.

The classical approach of studying the formation of coalitions assumes a transferable utility (TU)-framework and is based on the characteristic function. This function assigns to every coalition a worth, which is the aggregate payoff a coalition can secure for its members, irrespective of the behavior of players outside this coalition. The focus of this approach is on the coalition's worth in the grand coalition and how it might be shared among all players. However, smaller coalitions and the payoff to outsiders are typically neglected. Hence, by construction, the characteristic function approach appears to be ill-suited for the analysis of externality problems in which outsiders' actions affect the worth of coalitions and vice versa (Bloch 2003). An alternative approach is the partition function introduced by Thrall and Lucas (1963). This function assigns a worth to every coalition and singleton in the game. The worth depends on the entire coalition structure, *i.e.* the partition of players. The main focus of this literature is on the prediction of equilibrium coalition structures emerging from some coalition formation process. Since in our analysis externalities are a central feature, this paper is based on the partition function.

Beyond the particularities of the underlying economic models, it appears that games in partition function form share some common features in terms of the incentives to form coalitions. As extensively discussed for instance by Yi (1997), Bloch (2003) and Maskin (2003), a crucial feature that determines the success of coalition formation is the sign of the externality. In positive (negative) externality games, players not involved in the enlargement of coalitions are better (worse) off through such a move. Hence, in positive externalities games, typically, only small coalitions are stable, as players have an incentive to stay outside coalitions. Typical examples of positive externalities include output and price cartels and the provision of public goods. Firms not involved in an output cartel benefit from lower output by the cartel via higher market prices. This is also the driving force in price cartels where the cartel raises prices above non-cooperative levels. Similarly, agents not involved in a public good agreement benefit from higher provision levels of participants. Given the fact that cooperation proves usually difficult in positive externality games, we will mainly focus on this class of games. In contrast in negative externality games, outsiders have an incentive to join coalitions and therefore most coalition models predict the grand coalition as a stable outcome. Examples include for instance trade agreements which impose tariffs on imports from outsiders or R&D-collaboration among firms in imperfect markets where members gain a comparative advantage over outsiders if the benefits from R&D accrue exclusively to coalition members. However, even in negative externality games cooperation is no longer trivial, once we depart from the assumption of symmetric players and hence the distribution of the gains from cooperation becomes crucial. It will become apparent that almost all our results carry directly over to negative externality games.

As just mentioned, regardless of the sign of the externality, it is intuitively clear that the success of cooperation depends crucially on the division of the gains from cooperation. This may explain why there is a substantial body of literature on the axiomatic underpinning of surplus sharing rules in the context of the partition function approach; see for instance Myerson (1977), Bolger (1989), de Clippel and Serrano (2008), Macho-Stadler, Pérez-Castrillo and Wettstein (2007) and Grabisch and Funaki (2008). However, this literature often assumes cooperation among all agents (*i.e.* grand coalition) to be stable per se, despite there are plenty of economic problems where it seems reasonable to expect that the grand coalition will not emerge because of strong free-rider incentives (e.g. Ray and Vohra 1999, Maskin 2003 and Hafalir 2007). In contrast, we consider stability as a necessary condition for sustainable cooperation. That is, coalitional stability is an equilibrium outcome of our sharing scheme, not an a priori assumption.

Another body of literature using the partition function focuses on stability as we do but has paid little attention to the division of the gains from cooperation. Due to the complexity of partition functions, most papers assume a fixed sharing rule. One set of papers, which comprises the bulk of the literature, assumes ex-ante symmetric players with equal sharing (e.g. Bloch 2003, Yi and Shin 2000 and Ray and Vohra 2001). This is mainly done for analytical tractability. However, symmetry is a strong assumption that is difficult to justify in most economic environments.
Another set of papers — mainly related to the analysis of public good agreements — allows for asymmetric players but makes kind of ad hoc assumptions about particular sharing rules, most of which are classical solution concepts or modifications of them (e.g. Nash Bargaining Solution or Shapley Value; Barrett 1997, Botteon and Carraro 1997 and Weikard, Finus and Altamirano-Cabrera 2006). Clearly, this approach is also not satisfactory for at least three reasons. First, for none of these sharing rules it is known whether an equilibrium coalition structure exists. Second, the prediction of equilibrium coalition structures is sensitive to the particular specification of the sharing rule. Third, it remains an open question whether there are other sharing rules that could perform better in terms of the aggregate worth, letting alone whether there is a sharing rule that is "optimal". With optimal we mean that a sharing scheme achieves

the maximum aggregate worth, subject to the condition that the underlying coalition structure is stable.

In order to close the research gap identified above, at least partially, we propose a new class of sharing rules, the Proportional Surplus Sharing Scheme, abbreviated PSSS, and have a closer look at one of its members, the Equal Surplus Sharing Scheme (ESSS). Our analysis is based on the well-known concept of internal and external stability introduced by d'Aspremont et al. (1983) in the context of cartel formation. Players are assumed to have the choice between remaining at the fringe (*i.e.* forming a singleton coalition) or joining the cartel (*i.e.* forming a non-trivial coalition). The cartel is called stable if no cartel member has an incentive to leave (internal stability) and no outsider has an incentive to join (external stability) the cartel. Admittedly, the focus on a single non-trivial coalition is a restriction. However, this comes with large benefits. First, the concept of internal and external stability allows us to interpret the set of stable coalitions as the set of Nash equilibria in a simple 0-1-announcement game of coalition formation. Second, and most important, we are able to derive all results under minimal structural assumptions. For instance, no assumption about the heterogeneity of players is required. Moreover, for many results we do not have to impose any assumption at all on the properties of the partition function and for few results only some very general and mild properties are required. Consequently, our results apply to a vast number of economic problems.

A central concept and starting point in our analysis is the notion of Potentially Internally Stable (PIS) coalition structures. Loosely speaking, PIS coalition structures generate sufficient surplus for coalition members in order to satisfy their free-rider claims. In the presence of externalities, this may only be possible for a subset of all possible coalition structures. Hence, in positive externality games, our sharing scheme does not necessarily lead to efficiency in the sense that the grand coalition forms.

It will become apparent that the structure of our sharing scheme resembles the Nash Bargaining solution but with threat points that depend on the coalition structure.[1] Every coalition members receives his threat point payoff plus a share of the surplus from cooperation which is the difference between the aggregate coalitional worth and the sum of members' free-rider payoffs. Shares are represented by weights with the PSSS allowing for any positive set of weights and ESSS assumes equal weights.

For the PSSS we show that an internally and externally stable coalition always exists. Even more importantly, among the set of PIS coalition structures, the coalition structure that generates the highest aggregate worth is stable in positive externality games. In negative externality games, a similar result holds, though slightly more restrictive assumptions are required.

[1]A similar idea in the specific context of international environmental agreements has recently been advanced by Fuentes-Albero and Rubio (2010), McGuinty (2007) and Weikard (2009).

Finally, and regardless of the type of externality (and any other structural assumptions), we demonstrate that the ESSS is characterized by three axioms, coalitional efficiency, additivity and anonymity, usually associated with cooperative solution concepts, e.g. the Shapley Value, adapted to the context of partition function games.

Therefore, the ESSS which we propose is close to the axiomatic surplus sharing literature, though without assuming that the grand coalition will always form. In other words, the ESSS bridges two strands of literature: the cooperative and axiomatic surplus sharing literature (e.g. Myerson 1977, Bolger 1989, de Clippel and Serrano 2008, Macho-Stadler, Pérez-Castrillo and Wettstein 2007, van den Brink and Funaki 2009) and the non-cooperative coalition formation literature (e.g. Bloch 2003, Ray and Vohra 2001, Maskin 2003 and Faigle and Grabish 2011).

In the following, we first introduce notations and definitions in section 2. Section 3 introduces our new sharing scheme and proves its properties. The axiomatization of this sharing scheme for equal weights is the subject of section 4. Finally, section 5 summarizes the main findings and points to some issues of future research.

2 Ingredients

2.1 Partition Function

We denote by $N = \{1, ..., n\}$, $n \geq 2$, the set of players in the game. We consider coalition structures, *i.e.* partitions of the set of players N, comprising one non-empty coalition $S \subseteq N$ with $\#S = s > 0$ (the cartel or coalition) while all other players $j \in N \setminus S$ are singletons (the fringe). Let $\mathcal{B}$ be the set of all possible coalition structures of this type, $\mathcal{B} = \{\beta_S = (S, \{j_1\}, ..., \{j_{n-s}\}),\ S \subseteq N,\ j_1, ..., j_{n-s} \in N \setminus S\}$. We define a partition function π that assigns a single real number $\pi_S(\beta_S)$ to coalition S and real numbers $\pi_{\{j\}}(\beta_S)$ to every singleton $j \in N \setminus S$ of the fringe as follows:

$$\pi : \beta_S \in \mathcal{B} \mapsto \pi(\beta_S) = (\pi_S(\beta_S), \pi_{\{j_1\}}(\beta_S), ..., \pi_{\{j_{n-s}\}}(\beta_S)).$$

For notational simplicity, for any $j \in N$, we denote $\pi_{\{j\}}(\beta_S)$ by $\pi_j(\beta_S)$ in the sequel. The image of this mapping is a vector of variable size $1 + (n - s)$, depending on the cardinality of coalition S and on the total number of players. On the one hand, our partition function is simpler than general partition functions (see, e.g., Bloch 2003 and Yi 2003) since we disregard all partitions that consist of two or more non-trivial coalitions. On the other hand, and in contrast to the classic characteristic function, our partition function assigns not only a worth to coalition S but also to the non-members of S. This is important because information on the payoffs to players outside the coalition is indispensable for analyzing the incentives to leave or join a coalition in games with externalities.

We define a transferable utility partition function game $\Gamma = (N, \mathcal{B}, \pi)$, with the set of players $N = \{1, ..., n\}$ (with $n \geq 2$), a set of partitions or coalition structures $\mathcal{B}$ and a partition function π mapping to each coalition structure $\beta_S \in \mathcal{B}$ a real valued vector, representing the worth of all the coalitions in β_S. Following, among others, Yi and Shin (2000), Maskin (2003) and Hafalir (2007), we introduce a general property of partition functions that proves useful in grouping externalities in two broad classes.

Definition 1 *: **Positive and Negative Externalities***

A game in partition function form $\Gamma(N, \mathcal{B}, \pi)$ exhibits positive (negative) externalities if and only if its partition function π satisfies: $\forall S \subseteq N, \forall j \in N \setminus S, j \neq i : \pi_j(\beta_S) \geq (\leq) \pi_j(\beta_{S\setminus\{i\}})$ and $\exists k \in N \setminus S, k \neq i : \pi_k(\beta_S) > (<) \pi_k(\beta_{S\setminus\{i\}})$.

Positive externalities imply that no outsider, *i.e.* a player that is not involved in the enlargement of a coalition, is worse off and at least one outsider is strictly better off whenever a a singleton joins coalition S. The opposite holds for a negative externalities. Note that a game is called a positive (negative) externality game if this property holds for all possible coalition structures in $\beta_S \in \mathcal{B}$.

Apart from this broad classification, it will turn out to be useful to refer sometimes to two additional properties.

Definition 2 *: **Superadditivity***

A game in partition function form $\Gamma(N, \mathcal{B}, \pi)$ is superadditive if and only if its partition function π satisfies: $\pi_S(\beta_S) \geq \pi_{S\setminus\{i\}}(\beta_{S\setminus\{i\}}) + \pi_i(\beta_{S\setminus\{i\}})$ for all $\beta_S \in \mathcal{B}$ and for all $i \in S$.

Superadditivity means that the worth of those players involved in a merger will not decrease. In positive externalities games, superadditivity is a sufficient condition that global welfare (*i.e.* the sum of worth over all players) increases when coalitions become gradually larger, a property also called full cohesiveness (see e.g. Montero 2006). It is worthwhile to recall that despite superadditivity the grand coalition may not be stable if the positive externality effect is stronger than the superadtivity effect (see for instance the example in the Appendix which we introduce below). For negative externality games, superadditivity does not necessarily imply that global welfare increases with the enlargement of coalitions, though one should expect that the grand coalition generates the highest global welfare as argued below.

Superadditivity is often motivated by arguing that "if two coalitions merge, they always have the option of behaving as they did when they were separate, and so their total payoff should not fall" (Maskin 2003, p. 9). However, this argument is not innocuous in partition function games if the economic strategies of players not involved in a merger do not remain fixed (Bloch 2003). In fact, in the presence of externalities, economic strategies may be strategic substitutes, like in price and output cartels, and hence superadditivity may either fail completely or may not hold for all coalition structures. This is well-know from the industrial economics literature (e.g. Salant, Switzer and Reynolds 1983) where not even a two-player cartel may be profitable and hence not stable. Despite the critical remark about the assumption of superadditivity, one may argue that in the context of negative externality games the analysis of coalition formation looses its normative appeal if superadditivity fails.

In contrast, a less innocuous and almost natural property in externality games is the property of cohesiveness, see Montero (2006) and Cornet (1998). Cohesiveness implies that the grand coalition generates the highest global welfare among all possible coalition structures.

Definition 3 *: **Cohesiveness***

A game in partition function form $\Gamma(N, \mathcal{B}, \pi)$ is cohesive if and only if its partition function π satisfies: $\pi_N(\beta_N) \geq \pi_S(\beta_S) + \sum_{j \notin S} \pi_j(\beta_S)$ for all $\beta_S \in \mathcal{B}$.

By the definition of an externality (regardless whether they are positive or negative), economic strategies (e.g. output, prices, tariffs and R&D investment) of at least one player have an impact on at least one other player. Consequently, the grand coalition can always internalize externalities across all players, and hence the total worth should not be lower than in any other coalition structure.

Note finally that we will make use of the above properties only in a few instances and then this will be mentioned explicitly. By default, we make no assumptions at all.

2.2 Valuation Function

For the analysis of the incentives of individual players to form coalitions, we have to take one more step: it is not the aggregate payoff to a coalition but the individual payoffs to coalition members that matter. This type of information is part of the *valuation function*, *i.e.* a function mapping coalition structures into a vector of individual payoffs, called valuations. We define a valuation function that assigns to every coalition structure $\beta_S \in \mathcal{B}$ a real-valued vector $f^\pi(\beta_S) \in \mathbb{R}^n$ as follows:

$$\begin{cases} \sum_{i \in S} f_i^\pi(\beta_S) = \pi_S(\beta_S) & \\ f_j^\pi(\beta_S) = \pi_j(\beta_S) & \text{if } j \notin S. \end{cases}$$

That is, for every coalition S, the valuation function f^π specifies how the worth of coalition S is distributed among its members. By construction, valuations are group rational, *i.e.* the entire worth $\pi_S(\beta_S)$ of coalition S is distributed among its members. For every outsider to coalition S, the valuation coincides with the worth $\pi_j(\beta_S)$ that is assigned to him by the partition function.

Obviously, there are many possibilities to construct valuation functions starting from a particular partition function. In the sequel, we will propose a particular class of valuation functions with some desirable properties for games with externalities.

Note the difference between the concept of a valuation and an imputation known from the characteristic function. An imputation is usually only one vector of length n, listing the payoff of each player in the grand coalition ($S = N$) whereas a valuation function assigns vectors of length n to every possible coalition structure, listing individual payoffs not only of coalition members but also of outsiders.[2] This more comprehensive view is necessary to capture externalities across coalitions and players.

Equipped with the definition of a valuation function, we can now introduce the notion of stable coalitions following d'Aspremont et al. (1983).

Definition 4 *: **Internal and External Stability***

Let f^π be a valuation function for the game in partition function form $\Gamma(N, \mathcal{B}, \pi)$ and $f(\beta_S)$ the vector of valuations of the players in N when coalition structure β_S forms. Coalition structure $\beta_S \in \mathcal{B}$ is stable *with respect to the valuation function f^π if and only if:*

$$\textit{internal stability: } \forall i \in S : f_i^\pi(\beta_S) \geq f_i^\pi(\beta_{S \setminus \{i\}}),$$

$$\textit{external stability: } \forall j \in N \setminus S : f_j^\pi(\beta_S) \geq f_j^\pi(\beta_{S \cup \{j\}}).$$

That is, coalition structure β_S is stable if it is internally (IS) and externally (ES) stable, *i.e.* no insider wants to leave and no outsider wants to join coalition S.

As there are many ways to share the coalitional worth in a coalition game Γ, there are as many possible valuation functions that can be derived from its partition function. Consequently, a partition β_S may be stable with respect to a particular valuation function f^π but may not be stable with respect to another valuation function g^π. Therefore, we denote the set of coalition structures that are internally stable with respect to valuation function f^π by $\Sigma^{IS}(f^\pi)$, the set of coalition structures that are externally stable by $\Sigma^{ES}(f^\pi)$ and the set of stable coalition structures by $\Sigma^S(f^\pi) = \Sigma^{IS}(f^\pi) \cap \Sigma^{ES}(f^\pi)$.

[2]An exception is Aumann and Drèze (1974) who consider various solution concepts not only for the grand coalition but also for any partition of players. We will apply one of their concepts to the example we introduce below.

In the following, we start our analysis by considering first the incentive of coalition members to leave coalition S,*i.e.*internal stability, and subsequently adding the dimension of external stability. This is because our new sharing scheme is mainly developed to remedy free-riding in the context of positive externality games, though we will show later that it is also useful in negative externality games. In positive externality games, it appears that one is more concerned about players leaving the coalition than about players joining it. Hence, the most immediate notion of free-riding seems to be related to the violation of internal stability, although we have to be aware that stability also comprises external stability. For this purpose, it will prove useful to classify coalition structures according to the following criterion.

Definition 5 *: **Potential Internal Stability***

Consider a game in partition function form $\Gamma = (N, \mathcal{B}, \pi)$. A coalition structure $\beta_S \in \mathcal{B}$ is called Potentially Internally Stable for game Γ if and only if:

$$\pi_S(\beta_S) \geq \sum_{i \in S} \pi_i(\beta_{S \setminus \{i\}}).$$

That is, a coalition structure is Potentially Internally Stable (PIS) if the aggregate payoff to the coalition is not smaller than the sum of the free-rider payoffs of its members. The free-riding payoff is the payoff that a player can achieve if he leaves coalition S and the remaining members continue cooperating in $S \setminus \{i\}$. Note that the property of PIS refers to the partition function, as only aggregate payoffs matter, whereas internal stability and external stability are (and have to be) properties of a valuation function, as individual payoffs matter. For later reference, we will denote the set of coalitions that are PIS for a particular partition function by π by $\Sigma^{PIS}(\pi)$.

3 Proportional Surplus Sharing Scheme

3.1 Motivation

In this section, we introduce our new surplus sharing scheme for partition function games with externalities, the Proportional Surplus Sharing Scheme (PSSS). Since it is our objective to study the impact of surplus sharing rules on the stability of coalitions in a general framework, the use of a specific sharing rule would be too restrictive. Therefore, we introduce a class of sharing rules and study the properties of its members. Recall that there corresponds to every surplus sharing rule a particular valuation function. Hence, speaking about a sharing rule is equivalent to speaking about a valuation function and it is the latter terminology that we will use in the remainder of this paper.

In order to illustrate some of our results in the course of the following discussion, we provide a numerical example in the Appendix. The example assumes four players and a partition function that exhibits positive externalities, see Table A.1 in the Appendix. As can be seen from column (10), all coalition structures comprising two-player coalitions, and two out of four coalition structures comprising a three-player coalition are PIS. The other two three-player coalitions and the grand coalition do not generate enough surplus to compensate for the free-riding claims of their members, despite the partition function in this example is superadditive. Consequently, the challenge for a surplus sharing rule is to achieve maximal global welfare and to ensure stability of the coalition structure at the same time.

Meeting this challenge is far from obvious as this is illustrated in Table A.2. In this table valuations of the Shapley Value are reported as defined in Aumann and Drèze (1974), *i.e.* these values are computed not only for the grand coalition but for every non-empty coalition. Two interesting points can be observed in this example. First, a stable coalition may not exist. Second, there is no indication as to whether there are other sharing rules leading to stable coalitions and in particular which sharing rule would imply the highest aggregate worth. In the following, we claim that there is such a sharing rule: the PSSS.

3.2 Definitions

The construction of the PSSS starts from the observation that a necessary and sufficient condition for internal stability of coalition structure β_S is that each player in S receives his outside payoff. In the cartel formation game, the free-rider payoffs are associated with the scenario in which an individual coalition member leaves coalition S in order to become a singleton while the remaining members of coalition S continue to cooperate. These payoffs constitute lower bounds on the claims of individual coalition members with respect to the coalitional surplus in order to refrain from leaving the coalition.

Definition 6 : ***Proportional Surplus Sharing Valuation Function***

A Proportional Surplus Valuation Function for a game in partition function form $\Gamma = (N, \mathcal{B}, \pi)$ *is a valuation function* v^π *that satisfies* $\forall S \subseteq N$*:*

$$\begin{cases} v_i^\pi(\beta_S) = \pi_i(\beta_{S\setminus\{i\}}) + \lambda_i(\beta_S)\sigma(\beta_S) \quad \forall i \in S \\ v_j^\pi(\beta_S) = \pi_j(\beta_S) \quad \forall j \in N \setminus S \end{cases}$$

with $\lambda(\beta_S) \in \Delta^{s-1} = \{\lambda \in \mathbb{R}^s_+ \mid \sum_{i\in S} \lambda_i = 1\}$ *and* $\sigma(\beta_S) = \pi_S(\beta_S) - \sum_{i\in S} \pi_i(\beta_{S\setminus\{i\}})$ *where* Δ^{s-1} *denotes the set of all possible positive sharing weights of a coalition with s players and* $\sigma(\beta_S)$ *denotes the surplus or deficit of coalition* S *over the sum of free-rider payoffs* $\pi_i(\beta_{S\setminus\{i\}})$ *of its members.*

Intuitively, the Proportional Surplus Sharing Valuation Function (PSSVF) allocates to each coalition member his free-rider payoff, plus a non-negative share of the remaining surplus (or deficit) in proportion to weights $\lambda(\beta_S)$. Differences in valuations of players can be the result of (i) different free-rider payoffs, and/or (ii) different weights according to which the surplus or deficit is shared.[3]

One obvious interpretation of our definition is that the free-rider payoff $\pi_i(\beta_{S\setminus\{i\}})$ is the threat point of player i in coalition S and weight $\lambda_i(\beta_S)$ is his bargaining power. Hence, the PSSVF can be seen as an extension of a Nash bargaining solution in TU-games in the context of games in partition function form. However, different from the Nash bargaining solution, the threat point is not fixed but depends on the coalition structure. For instance, if S gradually increases through the accession of players, the threat point will also gradually increase (decrease) as a result of positive (negative) externalities.

It is important to point out that there are as many PSSVFs for a game in partition function form as there are ways to share — in every possible coalition S of N — the coalitional surplus among its members. The set of all PSSVFs for game Γ will be denoted by $\mathcal{V}(\Gamma)$ and constitutes the PSSS. Despite that every PSSVF is defined for specific weights λ, the class of PSSVFs, constituting the PSSS, does not require assumptions about weights as long as they are non-negative and sum up to one for each coalition S of N. This stresses that — different from most cooperative solution concepts, *e.g.*, Shapley Value and Nucleolus — the PSSS does not require to assign a unique value to each player in some coalition S. Instead, we find it more appealing that weights do not matter for many of our subsequent results, *i.e.* several results hold for the entire class of PSSVFs.

[3]Note the following link to characteristic function form games. For a given partition function form game $\Gamma = (N, \mathcal{B}, \pi)$, we can define a characteristic form game (N, u) with $u(S) = \pi_S(\beta_S)$ for any $S \subseteq N$. Assume that in the game Γ for any $S \subseteq N$ and any $j \notin S$ the value $\pi_j(\beta_S)$ does not depend on S. Then we can write the PSSVF for $S = N$ and equal weights as follows. For each $i \in N$, we have

$$v_i^\pi(\beta_N) = \pi_i(\beta_{N\setminus\{i\}}) + \frac{1}{n}\big[\pi_N(\beta_N) - \sum_{i\in N} \pi_i(\beta_{N\setminus\{i\}})\big] =$$

$$u(\{i\}) + \frac{1}{n}\big[u(N) - \sum_{i\in N} u(\{i\})\big] = CIS_i(N, u)$$

with $CIS_i(N, u)$ being the CIS-value for a characteristic form game as defined by Driessen and Funaki (1991). See also van den Brink and Funaki (2009) on this point.

Table A.3 displays results in the numerical example for our PSSVF. The PSSVF achieves internal stability of all nine PIS coalitions (*i.e.* compare Table A.1, column 10 with Table A.3, column 7). This observation will be generalized in Lemma 1 below. In contrast, the Shapley Value can only ensure internal stability of the singleton coalition structure and of all two-player coalitions (Table A.2, column 7) and hence misses to internally stabilize two potentially internally stable coalitions with three members.

3.3 Preliminary Properties

The following Lemma formalizes and generalizes the observations from above.

Lemma 1 : ***Potential Internal Stability of PSSVFs***

Consider a partition function form game $\Gamma(N, \mathcal{B}, \pi)$. Partition $\beta_S \in \mathcal{B}$ is potentially internally stable if and only if it is internally stable for any $v^\pi \in \mathcal{V}(\Gamma)$.

Proof:
$\Longrightarrow$ Suppose that coalition structure $\beta_S \in \mathcal{B}$ is PIS, implying $\sigma(\beta_S) \geq 0$, but assume to the contrary that there exists a valuation $v^\pi \in \mathcal{V}(\Gamma)$ such that partition $\beta_S \in \mathcal{B}$ would not be internally stable with respect to this valuation function. Thus, $\pi_i(\beta_{S\setminus\{i\}}) + \lambda_i(\beta_S)\sigma(\beta_S) = v_i^\pi(\beta_S) < v_i^\pi(\beta_{S\setminus\{i\}}) = \pi_i(\beta_{S\setminus\{i\}})$ which would imply $\sigma(\beta_S) < 0$ and therefore contradicts the initial assumption.

$\Longleftarrow$ If β_S is internally stable for any valuation $v^\pi \in \mathcal{V}(\Gamma)$, then it follows that $v_i^\pi(\beta_S) \geq v_i^\pi(\beta_{S\setminus\{i\}}) = \pi_i(\beta_{S\setminus\{i\}})$. Taking sums over all members in S and using the definition of PSSVF yields $\sum_{j\in S}\left[\pi_j(\beta_{S\setminus\{j\}}) + \lambda_j(\beta_S)\sigma(\beta_S)\right] \geq \sum_{j\in S}\pi_j(\beta_{S\setminus\{j\}})$ and hence $\sigma(\beta_S) \geq 0$. From Definition 6, it therefore follows that β_S is PIS. $\square$

The importance of Lemma 1 derives from three facts. First, internal stability is a necessary condition for stable coalitions, but is often violated for larger coalitions, in particular in positive externality games, as our example illustrates and as it appears from the literature (e.g. Maskin 2003 and Hafalir 2007). Second, every PSSVF ensures that every coalition structure that is PIS will actually be internally stable. Other solution concepts may miss this potential substantially as was observed in the numerical example. Third, there is a high degree of freedom in the choice of the sharing rule (through the choice of weights λ) when aiming at stabilizing coalitions internally.

The next Lemma 2 will turn out to be very useful in the sequel because it establishes an important link between internal and external stability for every member of the PSSS.

Lemma 2 *: **External and Potential Internal Stability of PSSVFs***

Consider a game in partition function form $\Gamma(N, \mathcal{B}, \pi)$ and a valuation function $v^\pi \in \mathcal{V}(\Gamma)$. If coalition structure β_S is not externally stable with respect to v^π, then there exists a player $j \in N \setminus S$ such that coalition structure $\beta_{S\cup\{j\}}$ is potentially internally stable.

Proof:
If coalition structure β_S is not externally stable with respect to v^π, then it follows from Definition 4 that: $\exists j \in N \setminus S : v_j^\pi(\beta_{S\cup\{j\}}) > v_j^\pi(\beta_S)$. Using Definition 6 of PSSVF this is equivalent to $\pi_j(\beta_S) + \lambda_j(\beta_{S\cup\{j\}})\sigma(\beta_{S\cup\{j\}}) > \pi_j(\beta_S)$ or $\sigma(\beta_{S\cup\{j\}}) > 0$, implying that $S \cup \{j\}$ is PIS. $\square$

It is important to note that Lemma 2 is a distinctive property of the class of PSSVFs. It may not hold for other valuation functions, for instance valuation functions based on traditional cooperative solutions, like the Shapley value, as this is evident from the numerical example in the appendix (see Table A.2).

Using Lemma 1 and 2, we now establish our first main result, the existence of a stable coalition structure for every PSSVF.

Proposition 1 *: **Existence of a Stable Coalition Structure***

Consider a game in partition function form $\Gamma(N, \mathcal{B}, \pi)$ and the corresponding class of PSSVFs $\mathcal{V}(\Gamma)$. For any $v^\pi \in \mathcal{V}(\Gamma)$ there exists at least one stable coalition structure $\beta_S \in \mathcal{B}$.

Proof:
By definition, the trivial coalition structure $(\{1\}, ..., \{n\})$ is internally stable. If it is also externally stable, we are done. Suppose, however, that this is not the case. Then there exists at least one two-player coalition that is PIS by Lemma 2 and which is internally stable for any $v^\pi \in \mathcal{V}(\Gamma)$ by Lemma 1. Again, if one of these two-player coalitions is also externally stable, we are done. Continuing with this reasoning, it is evident that some coalition $S \subseteq N$ will be eventually internally and externally stable, noting that $S = N$ is externally stable by definition. $\square$

It should be noted that the line of reasoning in the proof above follows closely the arguments developed in d'Aspremont et al. (1983). However, there is an important difference: we neither have to assume symmetric valuations for all players in S, *i.e.* $v_i^\pi(\beta_S) = v_j^\pi(\beta_S)\ \forall i, j \in S$, nor symmetric valuations for all non-member, *i.e.*

$v_k^\pi(S) = v_l^\pi(S)\ \forall k, l \in N \setminus S$, as this is done in d'Aspremont et al. (1983) and in much of the literature (see, e.g. Bloch 2003 and Yi 1997). Note also that no structural assumption about the underlying economic model is required and hence also not about the properties of the partition function game.

From the proof of Proposition 1 it is evident that it would be sufficient if there exists at least one non-trivial (*i.e.* a two-player or possibly larger) PIS coalition to ensure existence of a non-trivial stable coalition structure. In this case, we would have a starting point for applying the "algorithmic existence proof" above. In many economic examples such non-trivial PIS coalitions do exist. At a more general level, superadditivity would be a sufficient though not a necessary condition for the existence of a non-trivial stable coalition under the PSSS, as shown in the following corollary.

Corollary 1 : ***Existence of a Unique Stable Coalition Structure***

Consider a game in partition function form $\Gamma(N, \mathcal{B}, \pi)$ and the corresponding class of PSSVFs $\mathcal{V}(\Gamma)$. For any $v^\pi \in \mathcal{V}(\Gamma)$ there exists at least one stable non-trivial coalition structure $\beta_S \in \mathcal{B}$ if partition function π is superadditive.

Proof:
As result of superadditivity, $\pi_{\{i,j\}}(\beta_{\{i,j\}}) \geq \pi_i(\beta_{\{i\}}) + \pi_j(\beta_{\{j\}})$ holds for all coalitions with two members which implies that they are PIS. Hence, a proof in the spirit of the proof of Proposition 1 can be constructed, except that the starting point is not the trivial coalition but a coalition with two players. □

3.4 Maximal Welfare Stable Coalition Structure

We now turn to one of our central results, which is related to welfare optimality. It states that adopting the PSSS guarantees that the coalition structure which generates the highest aggregate worth (*i.e.* the total payoff over all players) among all PIS coalition structures will not only be internally stable but also externally stable and therefore stable. The remarkable aspect of this result is that a sharing scheme that, at first sight, is designed to foster internal stability, is also capable of ensuring external stability for those coalitions that are most desirable in terms of aggregate welfare.

Proposition 2 : ***Maximal Welfare Stable Coalition Structure in Positive Externality Games***

Let $\Sigma^{PIS}(\pi)$ be the set of coalition structures that are potentially internally stable in partition function form game $\Gamma(N, \mathcal{B}, \pi)$ with positive externalities and let β_{S^*} be the coalition structure with the highest aggregate worth in $\Sigma^{PIS}(\pi)$: $\pi_{S^*}(\beta_{S^*}) + \sum_{j \in N \setminus S^*} \pi_j(\beta_{S^*}) \geq$

$\pi_S(\beta_S) + \sum_{j \in N \setminus S} \pi_j(\beta_S)$ *for all* $\beta_S \in \mathcal{B}$. *Then, every valuation function* $v^\pi \in \mathcal{V}(\Gamma)$ *will make coalition structure* β_{S^*} *both (i) internally and (ii) externally stable and, hence, (iii) stable.*

Proof:
(i) Follows from Lemma 1.

(ii) Assume to the contrary that $\beta_{S^*} \in \Sigma^{PIS}(\pi)$ would not be externally stable for some valuation function $v^\pi \in \mathcal{V}(\pi)$. Hence, it follows from Definition 4 that there is an outsider $j \in N \setminus S^*$ who would strictly gain from joining coalition S^*: $v_j^\pi(\beta_{S^* \cup \{j\}}) > v_j^\pi(\beta_{S^*})$. Using Definition 6, it follows that $\pi_j(\beta_{S^*}) + \lambda_j(\beta_{S^* \cup \{j\}})\sigma(\beta_{S^* \cup \{j\}}) > \pi_j(\beta_{S^*})$ and therefore we have that $\sigma(\beta_{S^* \cup \{j\}}) > 0$ which is equivalent to $\sum_{k \in S^* \cup \{j\}} \pi_k(\beta_{S^* \cup \{j\}}) > \sum_{k \in S^* \cup \{j\}} \pi_k(\beta_{S^*})$. Hence, members of coalition $S^* \cup \{j\}$ are strictly better off under $S^* \cup \{j\}$ than under S^* and $S^* \cup \{j\}$ is PIS. At the same time, we know from the positive externalities property that outsiders to $S^* \cup \{j\}$ are not worse off under $S^* \cup \{j\}$ than under S^*: $\sum_{k \in N \setminus (S^* \cup \{j\})} \pi_k(\beta_{S^* \cup \{j\}}) \geq \sum_{k \in N \setminus (S^* \cup \{j\})} \pi_k(\beta_{S^*})$. Combining the inequalities of members and non-members of coalition S it would hold that $\sum_{k \in N} \pi_k(\beta_{S^* \cup \{j\}}) > \sum_{k \in N} \pi_k(\beta_{S^*})$, contradicting the initial assumption that S^* generates the highest worth among all coalition structures that are PIS. (iii) Follows from Definition 4. $\square$

Proposition 2 can be interpreted as saying that we cannot do any better in terms of global welfare than adopting the PSSS if the agreement is required to satisfy stability in the sense of d'Aspremont et al. (1983). Note first that this result is very general since it only requires that the underlying game exhibits positive externalities. There is no need for further assumptions like superadditivity and hence this result applies to many economic problems. Second, since the PSSS is a collection of parametric valuation function, related to a specific set of sharing weights $\lambda(\beta_S), \forall \beta_S \in \mathcal{B}$, there remains considerable flexibility how to allocate the surplus of the coalition without jeopardizing optimality.

In the numerical example in the Appendix, the PSSVF stabilizes the three-player coalition $(\{a, b, c\}, \{d\})$ with the highest aggregate payoff among all potentially internally stable coalitions (*i.e.* $13\frac{1}{3}$), achieving 75.5% of the maximal aggregate payoff which would be obtained in the grand coalition (*i.e.* $17\frac{2}{3}$), see Table A.3. In contrast, in this example, the Shapley value (see Table A.2) leads to seven internally stable coalition structures of which none is externally stable and hence stable.

Finally, one might wonder whether the maximal welfare result in Proposition 2 carries over to negative externality games. The answer is affirmative, provided we impose additional conditions on the partition function. Apart from cohesiveness, which we argued above is a quite natural assumption in externality games, we need superadditivity to make coalition formation attractive at all. (See the discussion in section 2.1.)

Under these two conditions, it can be easily shown that the PSSS stipulates the grand coalition to be the unique stable and welfare maximizing coalition structure.

Proposition 3 *: **Maximal Welfare Stable Coalition Structure in Negative Externality Games***

Consider a partition function form game $\Gamma(N, \mathcal{B}, \pi)$ of which the partition function π exhibits negative externalities, superadditivity and cohesiveness. Then, every valuation function $v^\pi \in \mathcal{V}(\Gamma)$ will make the grand coalition N (i) the unique stable and (ii) welfare maximizing coalition structure.

Proof:
It is shown in Weikard (2009), Theorem 3 p. 583, that the grand coalition is the unique stable coalition in superadditive negative externality games. In addition, it follows trivially from cohesiveness that the grand coalition generates the highest global welfare. □

Recall that cohesiveness is a rather weak and natural condition in externality games, and certainly much weaker than full cohesivenss as discussed Cornet (2003) and Montero (2006).

4 Characterization

In this section, we show how our new sharing scheme, the PSSS, relates to existing cooperative solutions. In order to facilitate this comparison, we provide a characterization of the PSSS in the spirit of the Shapley value, referring to axioms like efficiency, anonymity and additivity. However, as these axioms are defined for cooperative games, we have to adapt them to the context of partition function games.

4.1 Coalitional Efficiency

Coalitional Efficiency (CE) requires that, for every possible coalition structure $\beta_S \in \mathcal{B}$, the value of the coalition S in coalition structure β_S is fully distributed among its members and that outsiders get their individual payoff.

Axiom 1 : ***Coalitional Efficiency***

Consider a game in partition function form $\Gamma(N, \mathcal{B}, \pi)$. A valuation function f^π is called coalitionally efficient if and only if for all $\beta_S \in \mathcal{B}$ it holds that

$$\begin{cases} \sum_{i\in S} f_i^\pi(\beta_S) = \pi_S(\beta_S) & \\ f_j^\pi(\beta_S) = \pi_j(\beta_S) & \text{for any } j \in N \setminus S. \end{cases}$$

This axiom is important as it ensures that no surplus is wasted by the sharing scheme. By construction, see Definition 6, the PSSS satisfies this axiom.

Proposition 4 : ***Coalitional Efficiency of the PSSS***

Consider a game in partition function form $\Gamma(N, \mathcal{B}, \pi)$. The PSSS satisfies the axiom of coalitional efficiency for any coalition structure $\beta_S \in \mathcal{B}$.

4.2 Additivity

A second frequently considered axiom for a value or solution is Additivity (AD). Consider two games with partition functions π^1 and π^2. Loosely speaking, additivity requires that the outcome of the combined game, characterized by the sum of the partition functions, $\pi^1 + \pi^2$, is the same as the sum of the separate outcomes of both games.

Axiom 2 : ***Additivity***

Consider two games in partition function form $\Gamma(N, \mathcal{B}, \pi^1)$ and $\Gamma(N, \mathcal{B}, \pi^2)$. A valuation function f^π is called additive if and only if for any coalition structure $\beta_S \in \mathcal{B}$ and for every $i \in N$, it holds that

$$f_i^{\pi^1+\pi^2}(\beta_S) = f_i^{\pi^1}(\beta_S) + f_i^{\pi^2}(\beta_S).$$

Additivity may be motivated in several ways. For example, we may interpret the partition function as an expected payoff. Then additivity is desirable because one can sum over the values in different states of the world. Moreover, in cost-sharing games in which agents share the cost of several services, additivity is desirable because the cost of a joint service should be the sum of the cost of separate services.[4]

[4]Cases where additivity does not hold are discussed for instance in Kolpin (1996).

Proposition 5 *: **Additivity of the PSSS***

The PSSS satisfies the axiom of additivity for any pair π^1 and π^2 of partition functions.

Proof:
Consider two partition functions π^1 and π^2. We define the sum of the partition functions by $\bar{\pi} = \pi^1 + \pi^2$ such that for all $\beta_S \in \mathcal{B}$ it holds that (i) $\bar{\pi}_S(\beta_S) = \pi^1_S(\beta_S) + \pi^2_S(\beta_S)$ and (ii) for all $j \notin S$ $\bar{\pi}_j(\beta_S) = \pi^1_j(\beta_S) + \pi^2_j(\beta_S)$. For any $i \in S$, we have:

$$\begin{aligned}
v_i^{\bar{\pi}}(\beta_S) &= \bar{\pi}_i(\beta_{S\setminus\{i\}}) + \lambda_i(\beta_S)\left[\bar{\pi}_S(\beta_S) - \sum_{j\in S}\bar{\pi}_j(\beta_{S\setminus\{j\}})\right] \\
&= \left[\pi_i^1(\beta_{S\setminus\{i\}}) + \pi_i^2(\beta_{S\setminus\{i\}})\right] + \\
&\qquad \lambda_i(\beta_S)\left[\left[\pi_S^1(\beta_S) + \pi_S^2(\beta_S)\right] - \sum_{j\in S}\left[\pi_j^1(\beta_{S\setminus\{j\}}) + \pi_j^2(\beta_{S\setminus\{j\}})\right]\right] \\
&= \left[\pi_i^1(\beta_{S\setminus\{i\}}) + \lambda_i(\beta_S)\left[\bar{\pi}_S^1(\beta_S) - \sum_{j\in S}\bar{\pi}_j^1(\beta_{S\setminus\{j\}})\right]\right] + \\
&\qquad \left[\pi_i^2(\beta_{S\setminus\{i\}}) + \lambda_i(\beta_S)\left[\bar{\pi}_S^2(\beta_S) - \sum_{j\in S}\bar{\pi}_j^2(\beta_{S\setminus\{j\}})\right]\right] \\
&= v_i^{\pi^1}(\beta_S) + v_i^{\pi^2}(\beta_S).
\end{aligned}$$

□

Note that this proof hinges on the assumption that the weights $\lambda_i(\beta_S)$ do not depend on the partition function π. However, they may depend on other parameters, like for instance the number of players in the game.

4.3 Anonymity

Anonymity (AN) requires that the outcome of the sharing scheme does not depend on the identity of players.

First note that this axiom requires a stronger adaptation to our context than the previous two axioms, coalitional efficiency and additivity, compared to their traditional definition in cooperative game theory. In particular, we cannot allow for any possible permutation of agents because this would upset the coalition structure. Therefore, we follow Aumann and Dreze (1974) and restrict permutations to respect to some extent the initial coalition structure. In particular, we allow only for permutations of players such that the coalition $S \in \beta_S$ remains intact. Second, we will turn to a specific member of the class of PSSS, in particular the Equal Surplus Sharing Scheme ESSS characterized by $\lambda_i = \lambda_j, \ \forall i, j \in N$.

Let ρ be a permutation of the set N, *i.e.* a one-to-one function from N to N. For any partition function π and any coalition structure β_S, consider a permutation ρ under which coalition S is invariant (*i.e.* for each $j \in S, \ \rho(j) = i \in S$). In this case, for each $\beta_S \in \mathcal{B}$ the corresponding coalition structure is $\rho(\beta_S) \in \mathcal{B}$, in particular $\rho(S) = S$. Also, define the composition $\rho\pi$ as follows:

$$\rho\pi[\rho(\beta_S)] = \pi(\beta_S).$$

Axiom 3 *: **Anonymity***

Consider a game in partition function form $\Gamma = (N, \mathcal{B}, \pi)$. A valuation function f^π satisfies anonymity if for any $i, j \in N$ such that $\rho(j) = i$ we have:

$$f_i^{\rho\pi}[\rho(\beta_S)] = f_j^\pi(\beta_S).$$

Anonymity implies that changing the order of players, while leaving the coalition structure intact, does not change the valuation of players. In fact, the outcome does depend on whether a player is an insider or outsider, but it does not depend on the exact position which a player takes among his group (*i.e.* group of coalition members or group of singletons).

Proposition 6 *: **Anonymity of the ESSS***

The ESSS (for any non empty coalition S with cardinality s, $\lambda_i(\beta_S) = 1/s$) satisfies anonymity.

Proof:
1) $i \in S, \rho(j) = i \Rightarrow j \in S$. So $\rho(S) = S$.

$$v_i^{\rho\pi}[\rho(\beta_S)] = \rho\pi_i(\beta_{S\setminus\{i\}}) + \frac{1}{s}\left[\rho\pi_S(\beta_S) - \sum_{i\in\rho(S)} \rho\pi_i(\beta_{S\setminus\{i\}})\right] =$$

$$\pi_j(\beta_{S\setminus\{j\}}) + \frac{1}{s}\left[\pi_S(\beta_S) - \sum_{j\in S} \pi_j(\beta_{S\setminus\{j\}})\right] = v_j^\pi(\beta_S).$$

2) $i \notin S$. We consider $\rho(j) = i$, so $j \notin S$.

$$v_i^{\rho\pi}[\rho(\beta_S)] = \rho\pi_i(\rho(\beta_S)) = \pi_j(\beta_S) = v_j^\pi(\beta_S)$$

□

4.4 Axiomatization

Proposition 7 : ***The ESSS is the unique solution satisfying CE, AD and AN***

Let $\Gamma(N, \mathcal{B}, \pi)$ be a game in partition function form. For any coalition structure $\beta_S \in \mathcal{B}$, the ESSS is the unique solution satisfying axioms coalitional efficiency, additivity and anonymity.

Proof:
Propositions 4-6 imply that for any coalition, the ESSS satisfies axioms coalitional efficiency, anonymity and additivity.[5]

We now prove that if another sharing scheme or solution concept for a game in partition function form $\Gamma(N, \mathcal{B}, \pi)$, say f^π, satisfies axioms CE, AD and AN, then it coincides with the ESSVF, *i.e.* the PSSVF with equal weights. The result is proved in the same spirit of Shapley's characterization theorem, *i.e.* by using the linearity of the characteristic form games space. For this, we introduce a technical definition, the so-called c-characteristic function. This allows us to specify the linear structure of the c-cooperative game space, *i.e.* the partition function form game space we are considering in this paper.

If $\mathcal{P}^*(N) = \{S \subseteq N, S \neq \emptyset\}$, given the partition function π, we will denote by π_c the *c-characteristic function* $\pi_c : \mathcal{P}^*(N) \times N \to \mathbb{R}$ defined for any $(S, i) \in \mathcal{P}^*(N) \times N$ as follows:

$$\pi_c(S,i) = \begin{cases} \pi_S(S,i) = \pi_S(\beta_S) & \text{if } i \in S, \\ \pi_i(S,i) = \pi_i(\beta_S) & \text{if } i \notin S. \end{cases}$$

Let us note that the c-characteristic function is constant on the set S. We call the pair (N, π_c) a c-cooperative game (which by definition coincides with the partition function form game $\Gamma = (N, \mathcal{B}, \pi)$) and denote by Π_c^n the space of all c-cooperative games with n players, *i.e.* the set of all possible π_c defined on $\mathcal{P}^*(N) \times N$. Let π_c^1 and π_c^2 be in Π_c^n and $\alpha \in \mathbb{R}$. Since for any $(S, i) \in \mathcal{P}^*(N) \times N$, $(\pi_c^1 + \pi_c^2)(S, i) = \pi_c^1(S, i) + \pi_c^2(S, i)$ and $(\alpha\pi_c^1)(S, i) = \alpha\pi_c^1(S, i)$, Π_c^n turns out to be a linear space.

The dimension of Π_c^n is $2^n - 1 + n2^{n-1} - n^2$.

We have one coalition structure of singletons $(\{1\}, ..., \{n\})$ and for every coalition S with $s \geq 2$ the number of possible coalition structures with a coalition of cardinality s is

$$\binom{n}{s}(n - s + 1)$$

[5] Note that the assumption of equal weights is only necessary for anonymity. The other properties, coalitional efficiency and addivity, hold for the more general class of the PSSS with arbitrary weights.

and hence the dimension is given by

$$n+\sum_{s=2}^{n}\left(\begin{array}{c}n\\ s\end{array}\right)(n-s+1)=n+\sum_{s=2}^{n}\left(\begin{array}{c}n\\ s\end{array}\right)+n\sum_{s=2}^{n-1}\left(\begin{array}{c}n-1\\ s\end{array}\right)=$$

$$2^n-1+n(2^{n-1}-n).$$

A basis is given by $\{\delta_T,\ T\subseteq N, T\neq\emptyset\}\cup\{\delta_{T,j},\ T\subset N, j\notin T\}$ where

$$\delta_T(S,i)=\begin{cases}1 & \text{if } T=S \text{ and } i\in T,\\ 0 & \text{otherwise.}\end{cases}$$

$$\delta_{T,j}(S,i)=\begin{cases}1 & \text{if } T=S \text{ and } i=j\notin T,\\ 0 & \text{otherwise.}\end{cases}.$$

Let $\pi\in\Pi_c^n$ be a c-characteristic function. Then for any $(S,i)\in\mathcal{P}^*(N)\times N$ we have

$$\pi_c(S,i)=\sum_{T\subset N}\{c_T\delta_T(S,i)+\sum_{j\notin T}c_{T,j}\delta_{T,j}(S,i)\}+c_N\delta_N(S,i)$$

with $c_T, c_{T,j}\in\mathbb{R}$.

Let $\Gamma=(N,\mathcal{B},\pi)$ and π be a partition function. For any coalition structure $\beta_S\in\mathcal{B}$ define π_c as the c-characteristic function associated with the partition function π for the game Γ. In the following, we will omit the subscript c, the argument will make it clear which function we consider. Let f^π be a solution concept for the partition function π, *i.e.*

$$f^\pi:\beta_S=(S,\{j_1\},...,\{j_{n-s}\})\in\mathcal{B}\mapsto(f_1^\pi(\beta_S),...,f_n^\pi(\beta_S))\in\mathbb{R}^n.$$

For any $\beta_S\in\mathcal{B}$, we can also consider the solution f^π defined on the set $\mathcal{P}^*(N)\times N$ as follows

$$f^\pi(S,i)=\begin{cases}f_i^\pi(\beta_S) & \text{if } i\in S,\\ f_j^\pi(\beta_S)=\pi_j(\beta_S) & \text{if } j\notin S.\end{cases}$$

Let us suppose that the solution concept f^π satisfies the axioms CE, AD and AN. By using AD, we have:

$$f^\pi=f^{c_N\delta_N}+\sum_{T\subset N}f^{c_T\delta_T+\sum_{j\notin T}c_{T,j}\delta_{T,j}}$$

and since also the ESSS ν^π satisfies additivity,

$$\nu^\pi=\nu^{c_N\delta_N}+\sum_{T\subset N}\nu^{c_T\delta_T+\sum_{j\notin T}c_{T,j}\delta_{T,j}}.$$

It is sufficient to prove that for any $(S,i) \in \mathcal{P}^*(N) \times N$, we have

$$f^{c_N \delta_N}(S,i) = \nu^{c_N \delta_N}(S,i) \tag{1}$$

and for any coalition $T \subset N$, for any $(S,i) \in \mathcal{P}^*(N) \times N$ and $j \notin T$, we have

$$f^{c_T \delta_T + \sum_{j \notin T} c_{T,j} \delta_{T,j}}(S,i) = \nu^{c_T \delta_T + \sum_{j \notin T} c_{T,j} \delta_{T,j}}(S,i). \tag{2}$$

Since for any $(S,i) \in \mathcal{P}^*(N) \times N$ the function $\delta_N(S-\{i\},i) = 0$ by definition of ESSS, we have

$$\nu^{c_N \delta_N}(S,i) = c_N \delta_N(S-\{i\},i) + 1/s[c_N \delta_N(S,i) - \sum_{i \in S} c_N \delta_N(S-\{i\},i)] = c_N 1/n$$

if $S = N$ and zero otherwise.

By CE, we have $\sum_{i \in S} f^{c_N \delta_N}(S,i) = c_N \delta_N(S,i) = c_N$ if $S = N$ and zero otherwise. Let us consider a permutation $\rho : N \to N$ such that $\rho(S) = S$ and $\rho(j) = i$; define $\rho c_N \delta_N(\rho(S),i) = c_N \delta_N(S,j)$. By AN, we have $f^{\rho c_N \delta_N}(\rho(N),i) = f^{c_N \delta_N}(N,j)$ so that for any $i,j \in N$ the values coincide $f^{c_N \delta_N}(N,i) = f^{c_N \delta_N}(N,j)$. This implies that $\sum_{i \in N} f^{c_N \delta_N}(N,i) = n f^{c_N \delta_N}(N,i) = c_N$. Equality (1) is therefore proven.

Fix now a coalition $T \subset N$. We have to prove equality (2) by distinguishing four cases.

2_a) $S = T, i = j \notin S = T$:

By definition of the ESSS, with $i \notin S$, we have $\nu^{c_S \delta_S + \sum_{i \notin S} c_{S,i} \delta_{S,i}}(S,i) = c_{S,i}$. But then, by CE, with $i \notin S$, we have

$$f^{c_T \delta_T}(S,i) + \sum_{j \notin T} f^{c_{T,j} \delta_{T,j}}(S,i) = c_T \delta_T(S,i) + \sum_{j \notin T} c_{T,j} \delta_{T,j}(S,i) =$$

$$0 + \sum_{j \notin S} c_{S,j} \delta_{S,j}(S,i) = c_{S,i}.$$

2_b) $S = T, i \in S, j \notin S = T$:

Since $i \in S$, $\delta_{T,j}(S,i) = 0$ and since $j \notin T$ by definition of the ESSS, we have that $\nu^{c_T \delta_T + \sum_{j \notin T} c_{T,j} \delta_{T,j}}(S,i) = \nu^{c_T \delta_T}(S,i) = c_S 1/s$. Analogously, $f^{c_T \delta_T + \sum_{j \notin T} c_{T,j} \delta_{T,j}}(S,i) = f^{c_S \delta_S}(S,i)$, and by CE we have $\sum_{i \in S} f^{c_S \delta_S}(S,i) = c_S \delta_S(S,i) = c_S$. By using AN, we have $f^{c_S \delta_S}(S,i) = f^{c_S \delta_S}(S,j)$. This implies that $\sum_{i \in S} f^{c_S \delta_S}(S,i) = s f^{c_S \delta_S}(S,i) = c_S$.

2_c) $T = S-\{i\}, i = j \in S, j \notin T$:

We prove that

$$f^{c_{S-\{i\}} \delta_{S-\{i\}} + \sum_{i \notin S-\{i\}} c_{S-\{i\},i} \delta_{S-\{i\},i}}(C,h) = \nu^{c_{S-\{i\}} \delta_{S-\{i\}} + \sum_{i \notin S-\{i\}} c_{S-\{i\},i} \delta_{S-\{i\},i}}(C,h)$$

for any $C \subseteq N$ and $h \in N$.

If $h \notin C$, by CE equality (2) is satisfied because both members are equal to

$$\gamma(C,h) = [c_{S-\{i\}}\delta_{S-\{i\}} + \sum_{i \notin S-\{i\}} c_{S-\{i\},i}\delta_{S-\{i\},i}](C,h).$$

For $h \in C$, the function $\gamma(C,h)$ is zero if $C \neq S-\{i\}$ and equality (2) holds because both members are 0. Let us suppose $C = S-\{i\}$ and $h \in S-\{i\}$. So

$$\nu^{\gamma}(S-\{i\},h) = \gamma_h(S-\{i\}-\{h\},h) + 1/(s-1)[\gamma_{S-\{i\}}(S-\{i\},h)-$$

$$\sum_{h \in S-\{i\}} \gamma_h(S-\{i\}-\{h\},h)] = c_{S-\{i\}}1/(s-1).$$

But then $f^{\gamma}(S-\{i\},h) = f^{c_{S-\{i\}}\delta_{S-\{i\}}}(S-\{i\},h)$. By (CE),

$$\sum_{h \in S-\{i\}} f^{c_{S-\{i\}}\delta_{S-\{i\}}}(S-\{i\},h) = c_{S-\{i\}}\delta_{S-\{i\}}(S-\{i\},h) = c_{S-\{i\}}$$

and by AN

$$f^{c_{S-\{i\}}\delta_{S-\{i\}}}(S-\{i\},h) = c_{S-\{i\}}1/(s-1).$$

$2_d)$ $T \neq S, T \neq S-\{i\}, i \in S \cap T, j \notin T$:

In this case, the function $\gamma(S,i) = [c_T\delta_T + \sum_{j \notin T} c_{T,j}\delta_{T,j}](S,i) = 0$, hence equality (2) holds because both members are 0. $\square$

In order to appreciate Proposition 7, it is worthwhile to recall (see Aumann and Dreže 1974) that the Shapley value in games in partition function form is characterized by relative efficiency, symmetry (anonymity), additivity and the null-player condition. The solution concept introduced in this paper requires only the first three axioms.

5 Conclusion

In this paper, we have introduced a new sharing scheme, the Proportional Surplus Sharing Scheme (PSSS) for the distribution of the gains from cooperation in games with externalities. In these games, the formation of the grand coalition cannot be taken for granted, which is particular true in the presence of positive externalities. Our analysis was based on games in partition function form in which coalition structures consist of one genuine coalition plus singletons. Particular attention was given to games which exhibit positive externalities from coalition formation, *i.e.* games in which outsiders to a coalition benefit from the enlargement of the coalition. However, most of our results hold irrespective of the type of externality and those derived for positive externalities carry over to negative externalities with minor modification. Prominent examples of positive externality games include the provision of public goods, like international cooperation between countries on issues of security, environment, customs or monetary unions, and cooperations between private companies on matters like R&D, procurement, output or price coordination. Examples of negative externality games are various forms of trade agreements which impose tariffs on imports from outsiders.

We showed that the PSSS achieves the maximum aggregate welfare subject to the constraint that coalitions have to be stable in the sense of d'Aspremont et al. (1983). More precisely, the PSSS attains the highest possible aggregate welfare among the set of coalitions that can potentially be internally stabilized. Potential internal stability (PIS) means that coalitions generate sufficient surplus to compensate for the free-riding claims of their members. The grand coalition may or may not be a member of this set of PIS coalitions, though in negative externality games it is the most likely outcome. Our sharing scheme is flexible because, irrespective of the weights attached to individual coalition members, the maximum welfare result holds.

An other important contribution of this paper was the complete characterization of a particular member of the PSSS. For the Equal Surplus Sharing Scheme (ESSS), which is a PSSS with equal sharing weights, it turns out that, much in the spirit of characterizations of the Shapley value in cooperative game theory, the axioms coalitional efficiency, additivity and anonymity hold.

Hence, we improved upon the existing literature on coalition formation in the context of externalities in several respects. We departed from the assumption of symmetric agents, established the existence of stable coalitions, characterized the solution concept in terms of well established axioms and showed optimality subject to stability.

For future research, two possible extensions seem obvious though difficult. First, our approach could be generalized to allow for the co-existence of several non-trivial coalitions. However, there is doubt, multiple coalitions would complicate the analysis tremendously because outside or threat point payoffs are not straightforwardly defined any longer as threat points are mutually depended and linked. Second, for many of our results, the PSSS leaves the choice of surplus sharing weights open. Endogenizing the value of these weights, which may be interpreted as bargaining power, in games with heterogeneous players and externalities seems an interesting topic for further research.

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Appendix

Table A.1 displays a four-player example with a partition function that exhibits positive externalities. The table displays the coalition structure (column 1), the worth to coalition S (column 2) and to non-members of S (column 3-6) and the aggregate worth to all players (column 7), the sum of free-rider payoffs (column 8; see Definition 5), the surplus to the coalition (column 9; see Definition 6) which is the difference between column 8 and 2 and indicates coalitions that are (not) potentially internally stable coalitions with 1 (0) (column 10).

We display valuations for the Shapley value in Table A.2. We use the extended version of the Shapley value as defined in Aumann and Drèze (1974), implying that these values are computed for every non-empty coalition and not only for the grand coalition. The values of the Shapley value are computed according to the following formula:

$$\forall \beta_S \in \mathcal{B},\ \forall i \in S: \qquad v_i^{Shp}(\beta_S) = \sum_{T \subseteq S} \frac{(t-1)!(s-t)!}{t!} \left[\pi_T(\beta_T) - \pi_{T\setminus\{i\}}(\beta_{T\setminus\{i\}})\right].$$

Table A.1: Partition Function

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
β_S	π_S	π_a	π_b	π_c	π_d	π_N	$\pi_S(\beta_{S\setminus\{j\}})$	$\sigma(\beta_S)$	Σ^{PIS}
$(\{a\},\{b\},\{c\},\{d\})$	-	0	0	0	0	0	-	-	1
$(\{a,b\},\{c\},\{d\})$	1	-	-	1	1	3	0	1	1
$(\{a,c\},\{b\},\{d\})$	1	-	$2\frac{1}{3}$	-	$3\frac{1}{3}$	$6\frac{2}{3}$	0	1	1
$(\{a,d\},\{b\},\{c\})$	$2\frac{1}{3}$	-	1	2	-	$5\frac{1}{3}$	0	$2\frac{1}{3}$	1
$(\{b,c\},\{a\},\{d\})$	3	$3\frac{1}{3}$	-	-	1	$7\frac{1}{3}$	0	3	1
$(\{b,d\},\{a\},\{c\})$	1	$3\frac{1}{3}$	-	2	-	$6\frac{1}{3}$	0	1	1
$(\{c,d\},\{a\},\{b\})$	1	1	6	-	-	8	0	1	1
$(\{a,b,c\},\{d\})$	9	-	-	-	$4\frac{1}{3}$	$13\frac{1}{3}$	$6\frac{2}{3}$	$2\frac{1}{3}$	1
$(\{a,b,d\},\{c\})$	$5\frac{1}{3}$	-	-	3	-	$8\frac{1}{3}$	$5\frac{1}{3}$	0	1
$(\{a,c,d\},\{b\})$	$5\frac{1}{3}$	-	7	-	-	$12\frac{1}{3}$	$6\frac{1}{3}$	-1	0
$(\{b,c,d\},\{a\})$	8	$4\frac{1}{3}$	-	-	-	$12\frac{1}{3}$	9	-1	0
$(\{a,b,c,d\})$	$17\frac{2}{3}$	-	-	-	-	$17\frac{2}{3}$	$18\frac{2}{3}$	-1	0

Table A.2: Valuation Function for the Shapley Value

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
β_S	v_a^{Shp}	v_b^{Shp}	v_c^{Shp}	v_d^{Shp}	v_N^{Shp}	Σ^{IS}	Σ^{ES}	Σ^{S}
$(\{a\},\{b\},\{c\},\{d\})$	0	0	0	0	0	1	0	0
$(\{a,b\},\{c\},\{d\})$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	3	1	0	0
$(\{a,c\},\{b\},\{d\})$	$\frac{1}{2}$	$2\frac{1}{3}$	$\frac{1}{2}$	$3\frac{1}{3}$	$6\frac{2}{3}$	1	0	0
$(\{a,d\},\{b\},\{c\})$	$1\frac{1}{6}$	1	2	$1\frac{1}{6}$	$5\frac{1}{3}$	1	0	0
$(\{b,c\},\{a\},\{d\})$	$3\frac{1}{3}$	$1\frac{1}{2}$	$1\frac{1}{2}$	1	$7\frac{1}{3}$	1	0	0
$(\{b,d\},\{a\},\{c\})$	$3\frac{1}{3}$	$\frac{1}{2}$	2	$\frac{1}{2}$	$6\frac{1}{3}$	1	0	0
$(\{c,d\},\{a\},\{b\})$	1	6	$\frac{1}{2}$	$\frac{1}{2}$	8	1	0	0
$(\{a,b,c\},\{d\})$	$2\frac{1}{3}$	$3\frac{1}{3}$	$3\frac{1}{3}$	$4\frac{1}{3}$	$13\frac{1}{3}$	0	1	0
$(\{a,b,d\},\{c\})$	2	$1\frac{1}{3}$	3	2	$8\frac{1}{3}$	0	0	0
$(\{a,c,d\},\{b\})$	2	7	$1\frac{1}{3}$	2	$12\frac{1}{3}$	0	1	0
$(\{b,c,d\},\{a\})$	$4\frac{1}{3}$	3	3	2	$12\frac{1}{3}$	0	1	0
$(\{a,b,c,d\})$	4	5	5	$3\frac{2}{3}$	$17\frac{2}{3}$	0	1	0

Valuations are computed applying the formula above to the partition function in Table A.1.

Table A.3: Valuation Function for the Proportional Surplus Sharing Scheme

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
β_S	v_a^{PSSS}	v_b^{PSSS}	v_c^{PSSS}	v_d^{PSSS}	v_N^{PSSS}	Σ^{IS}	Σ^{ES}	Σ^{S}
$(\{a\},\{b\},\{c\},\{d\})$	0	0	0	0	0	1	0	0
$(\{a,b\},\{c\},\{d\})$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	3	1	0	0
$(\{a,c\},\{b\},\{d\})$	$\frac{1}{2}$	$2\frac{1}{3}$	$\frac{1}{2}$	$3\frac{1}{3}$	$6\frac{2}{3}$	1	0	0
$(\{a,d\},\{b\},\{c\})$	$1\frac{1}{6}$	1	2	$1\frac{1}{6}$	$5\frac{1}{3}$	1	1	1
$(\{b,c\},\{a\},\{d\})$	$3\frac{1}{3}$	$1\frac{1}{2}$	$1\frac{1}{2}$	1	$7\frac{1}{3}$	1	0	0
$(\{b,d\},\{a\},\{c\})$	$3\frac{1}{3}$	$\frac{1}{2}$	2	$\frac{1}{2}$	$6\frac{1}{3}$	1	1	1
$(\{c,d\},\{a\},\{b\})$	1	6	$\frac{1}{2}$	$\frac{1}{2}$	8	1	1	1
$(\{a,b,c\},\{d\})$	$4\frac{1}{9}$	$3\frac{1}{9}$	$1\frac{7}{9}$	$4\frac{1}{3}$	$13\frac{1}{3}$	1	1	1
$(\{a,b,d\},\{c\})$	$3\frac{1}{3}$	1	3	1	$8\frac{1}{3}$	1	1	1
$(\{a,c,d\},\{b\})$	$\frac{2}{3}$	7	$1\frac{2}{3}$	3	$12\frac{1}{3}$	0	1	0
$(\{b,c,d\},\{a\})$	$4\frac{1}{3}$	$5\frac{2}{3}$	$1\frac{2}{3}$	$\frac{2}{3}$	$12\frac{1}{3}$	0	1	0
$(\{a,b,c,d\})$	$4\frac{1}{12}$	$6\frac{3}{4}$	$2\frac{3}{4}$	$4\frac{1}{12}$	$17\frac{2}{3}$	0	1	0

Valuations are computed applying Definition 6 to the partition function in Table A.1 and assuming equal weights, $\lambda_i(\beta_S) = \frac{1}{s}\ \forall i \in S$ and $\forall \beta_S$.