A ‘divide and choose’ approach to compromising*

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Abstract

We study dispute resolution in the compromise model of Börgers and Postl (2009), which provides an alternative framework for analyzing the real-world procedure of tri-offer arbitration studied in Ashenfelter et al (1992). Two parties involved in a dispute have to choose between their conflicting positions and a compromise settlement proposed by a neutral mediator. We ask how an adaptation of the familiar ‘divide and choose’ mechanism (DCM) performs as a protocol for dispute resolution in the absence of an arbitrator. We show that there is a unique equilibrium of the DCM if the parties’ von Neumann Morgenstern utilities from the compromise settlement are drawn independently from a concave distribution, or from any Beta-distribution (which need not be concave). Furthermore, for Beta-distributions that concentrate increasing probability mass on high von Neumann Morgenstern utilities of the compromise, the social choice rule implied by the DCM is asymptotically ex post Pareto efficient.

Keywords: arbitration; divide and choose; collective decision making; private information.

JEL classification: C72, D71, D82.

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1 Introduction

The study of different arbitration procedures for settling two-party conflicts occupies a prominent position in the literature on dispute resolution. These procedures differ with regard to the amount of discretion they allow the arbitrator in imposing a binding settlement on the two parties (Farmer and Pecorino, 2008). Two commonly used procedures are: conventional arbitration, where the arbitrator can impose any settlement he deems appropriate; and final offer arbitration, where the arbitrator must select one of the two parties' conflicting positions (see Ashenfelter et al (1992), Brams et al (1991), and chapter 3 in Brams (2003), for an overview of these procedures). A third procedure - one that curtails arbitrator freedom while still allowing for the possibility of a compromise settlement - is tri-offer arbitration. Under this procedure, which is used to resolve public sector labor disputes in Iowa (Ashenfelter et al, 1992), the arbitrator must select either one of the two parties' favored positions, or a compromise settlement proposed by a neutral mediator prior to the start of the arbitration process.

In this note, our focus is on the collective choice problem at the heart of tri-offer arbitration: should the two parties to the dispute choose a settlement favored by one of the parties, or should they select the compromise settlement? Our objective is to devise a protocol for dispute resolution that does not require the presence of an arbitrator, and which can be implemented by the parties themselves. One motivation is that with arbitration, there remains the issue of arbitrator selection regardless of the chosen arbitration procedure.[1] If the two parties must engage in some protocol for arbitrator selection, why not let them engage directly in a dispute resolution protocol? To address this question, we follow the approach to fair-division procedures in Brams and Taylor (1996) by assuming that any settlement emerging from such a protocol is binding on the two parties, just like the settlement imposed by an arbitrator would be.

A natural framework for studying tri-offer dispute resolution (both with and without an arbitrator) is the compromise setting of Börgers and Postl (2009). The mechanism design approach taken there offers a perspective on the arbitrator's decision problem that is different from, but complementary to the way arbitrator behavior is modeled in Ashenfelter et al (1992): there, the arbitrator has exclusive knowledge of what constitutes the true 'fair' settlement in the dispute, and he will choose whichever feasible settlement is closest to it. In contrast, Börgers and Postl (2009) assume that the two parties hold privately relevant information about their preferences, such as their respective von Neumann Morgenstern utilities derived from the neutral mediator's compromise settlement. Under arbitration, the onus is on the arbitrator to elicit this information truthfully.[2] The main impossibility result in Börgers and Postl (2009) implies that incentive compatible tri-offer arbitration generally fails to implement (ex post) Pareto efficient settlements.

We propose here a conflict resolution protocol for the compromise setting of Börgers and Postl (2009) that combines aspects of classic 'divide and choose' mechanisms with aspects of ultimatum bargaining.[3] Under our protocol, a proposer (selected randomly from among the two parties involved in the dispute) suggests a lottery involving only the two parties' respective favored settlements. If the other party (the 'responder') agrees to the proposed lottery, then his

[1] de Clippel et al (2012) study this problem, along with specific procedures for selecting an arbitrator.

[2] The idea of taking a mechanism design approach to arbitration in a two-party dispute when the arbitrator lacks information about the parties' preferences can be traced back to Rosenthal (1978).

[3] See Brams and Taylor (1996) for an overview of divide and choose mechanisms, under which one agent partitions a (possibly heterogeneous) good into two, while the other agent chooses whichever partition he wants. Divide and choose mechanisms have also been proposed for the dissolution of indivisible partnerships, with one partner proposing a price, and the other partner deciding whether to sell his share or purchase the other partner's share at that price. See, e.g., Morgan (2004).

favored settlement will be implemented with the probability specified by the proposer's lottery. If, instead, the responder rejects the proposed lottery, the compromise settlement previously suggested by the neutral mediator is implemented.

In what follows, we show that under our 'divide and choose' protocol, both proposer and responder have a unique type-contingent equilibrium strategy for a vast class of distributions of von Neumann Morgenstern utilities (namely for all concave distributions, as well as the entire class of Beta-distributions, which contains concave and convex distributions, in addition to distributions that change curvature from one to the other). We then investigate the performance of our protocol with respect to the ex ante expected welfare it generates. While under the uniform distribution our protocol is outperformed by a mechanism proposed in Börgers and Postl (2009) (the so called 'cropped triangle rule'), we show that for Beta-distributions which concentrate increasing amounts of probability mass on high realizations, our protocol converges to an (ex post) Pareto efficient settlement.

2 Model

2.1 Basic Setup

Two agents $i = 1,2$ must choose one alternative from the set $\{a_0, a_1, a_2\}$. Each agent i prefers alternative a_i over alternative a_0, and alternative a_0 over alternative a_{-i} (subscript $-i$ refers to the agent other than i). These ordinal preferences are common knowledge. We refer to alternative a_0 as the compromise because it is the middle-ranked alternative for both agents. Agent i's von Neumann Morgenstern utility function is $u_i : \{a_0, a_1, a_2\} \to \mathbb{R}$. Utilities are normalized so that $u_i(a_i) = 1$ and $u_i(a_{-i}) = 0$ for all i. These aspects of the von Neumann Morgenstern utility functions are common knowledge. For each agent i we denote by t_i the utility of the compromise $u_i(a_0)$. We refer to t_i as agent i's type. We assume that t_i is a random variable which is only observed by agent i. The agents' types are stochastically independent, and they are identically distributed with cumulative distribution function G. We assume that G has support $[0,1]$, that its derivative g is continuous, and that $g(t_i) > 0$ for all $t_i \in (0,1)$. The joint distribution of $t \equiv (t_1, t_2)$ is common knowledge among the agents. In most of the following, we assume that G is the parameterized Beta-distribution, which has density $g(t_i) = h(t_i)/B(\alpha, \beta)$ and cumulative distribution function $G(t_i) = H(t_i)/B(\alpha, \beta)$, where $h(t_i) \equiv t_i^{\alpha-1}(1-t_i)^{\beta-1}$, $H(t_i) \equiv \int_0^{t_i} \tau^{\alpha-1}(1-\tau)^{\beta-1} d\tau$, and $B(\alpha, \beta) \equiv H(1)$ for $\alpha, \beta > 0$.

2.2 Divide and Choose Mechanism (DCM)

We consider here an adaptation of the familiar 'divide and choose' mechanism as a way of making a collective choice in the compromise setting described above. The rules of this adapted DCM are as follows: one of the two agents is chosen randomly as the 'proposer'. Each agent has probability $1/2$ of being proposer. If agent i is selected as proposer, he suggests to the other agent a lottery over their respective favorite alternatives $\{a_i, a_{-i}\}$. That is, the proposer chooses a probability $p \in [0,1]$ with which the *other* agent's favorite alternative is chosen by the lottery. The other agent, the 'responder', then chooses between saying 'yes' or 'no'. If the responder says 'yes', then the responder's favorite alternative a_{-i} is implemented with probability p, and the proposer's favorite alternative a_i is chosen with probability $1-p$. If the responder says 'no', then the compromise a_0 is implemented.

2.3 First best mechanisms and welfare

In order to evaluate the performance of the DCM, we shall draw below on comparisons with *first best social choice rules*. A social choice rule (SCR) is a function that assigns to every pair of the agents' types a lottery over the set of alternatives. I.e. $f : [0,1]^2 \to \Delta(\{a_0, a_1, a_2\})$, $t \mapsto (f_0(t), f_1(t), f_2(t))$, where $f_i(t)$ $(i = 1,2)$ is the probability that agent i's favorite alternative is selected, and $f_0(t)$ is the probability that the compromise is selected. A SCR is *first best* if $t_1 + t_2 > 1 \Rightarrow\ f_0(t) = 1$ and $t_1 + t_2 < 1 \Rightarrow\ f_0(t) = 0$. In addition to a comparison of first best SCRs with the SCR implied by the DCM, we wish to compare the performance of these rules according to the ex ante expected social welfare they generate. Noting that the components $f_0(t)$, $f_1(t)$, and $f_2(t)$ of any SCR f sum up to 1 for all t, we can express as follows the ex ante welfare of f, given by the sum of the agents' ex ante expected utilities:

$$W \equiv 1 + \int_0^1 \left(\int_0^1 f_0(t)\,(t_1 + t_2 - 1)\,g(t_2)\,dt_2 \right) g(t_1)\,dt_1 \tag{1}$$

3 Results

3.1 Equilibrium

In this section, we characterize the equilibrium of the DCM:

Proposition 1. *For all concave distributions, and all Beta-distributions, the unique equilibrium of the DCM features the following strategies:*

(i) The proposer assigns probability $p(t_i)$ to the responder's favorite alternative where, for every $t_i \in [0,1]$, $p(t_i)$ is the unique value $p \in (0,1)$ that solves:

$$1 - p - \frac{G(p)}{g(p)} = t_i \tag{2}$$

(ii) The responder says 'yes' if $t_{-i} < p(t_i)$, and says 'no' otherwise.

Proof. The proof is by backward induction. Consider first the responder j. If p is the probability assigned by the proposer i to the responder's favorite alternative, then the responder should say 'yes' if $t_{-i} < p$. Otherwise, he should say 'no'. Given the probability distribution of t_{-i}, this implies that the responder will say 'yes' with probability $G(p)$.

Now consider the proposer's decision problem. If he offers a probability p for the responder's favorite alternative and the responder says 'yes', then the proposer's utility will be $1 - p$. If, instead, the responder says 'no', the proposer's utility will be t_i. Therefore, the proposer's expected utility is:

$$G(p)(1-p) + (1 - G(p))\,t_i \tag{3}$$

An optimal choice $p(t_i)$ (i.e. one that maximizes the proposer's expected utility) must satisfy the first-order necessary condition in (2), which is obtained by setting equal to zero the first derivative of expected utility in (3), and then dividing this equation by $g(p)$. It is easy to see that for concave distributions G, each proposer-type t_i has a unique optimal choice $p(t_i)$. This follows immediately from the fact that if $g'(p) < 0$ for all p, then the left-hand side of (2) is a decreasing function of p with derivative $-2 + G(p)g'(p)/g(p)^2$. However, concavity of G is only sufficient, but not necessary for uniqueness of the proposer's optimal choice. In order to show that uniqueness prevails for a vast class of distributions, we focus in the following on the case where G is the parameterized Beta-distribution. The reason is that this class includes both concave and convex distributions, as well as distributions whose curvature changes from one to the other.

In order to prove item (i) of Prop. 1 for the Beta-distribution, we denote by $\sigma(p|\alpha,\beta)$ the function on the left-hand side of (2), which is given by $1-p-H(p)/h(p)$ under the Beta-distribution. We can then express the necessary condition in (2) more succinctly as $\sigma(p|\alpha,\beta)=t_i$. The key component of the proof of item (i) is an investigation of the monotonicity properties of the function σ. Our findings, derived formally in the Appendix, are illustrated graphically in Fig. 1, which depicts representative graphs of σ for Beta-distributions with $\beta>1$, $\beta=1$, and $\beta<1$, resp. As the figure shows, σ has a unique root p_0 in the interval $(0,1)$, and is strictly decreasing on $[0,p_0]$.[4]

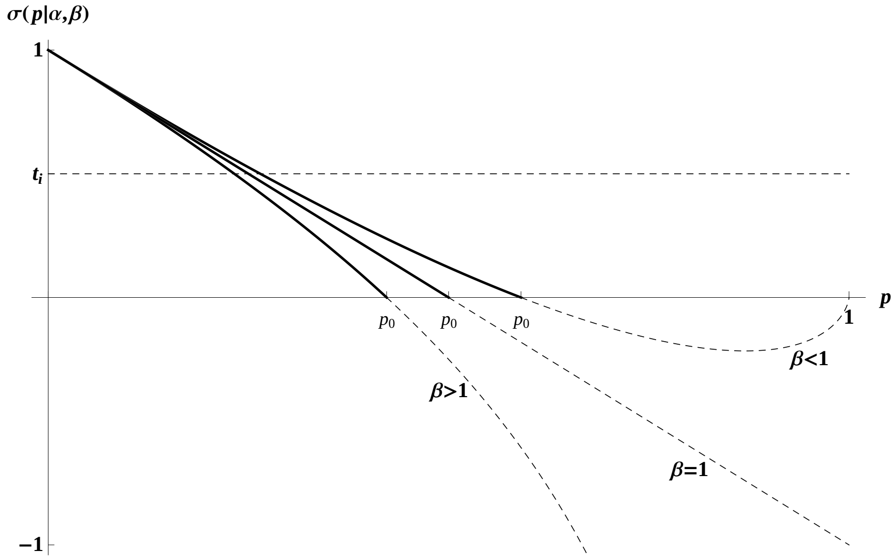

Figure 1: Representative graphs of σ for different Beta-distributions

It is obvious from Fig. 1 that for Beta-distributions with $\beta\geq 1$, every proposer-type $t_i\in[0,1]$ has a unique optimal choice $p(t_i)$. This is follows immediately from the fact that the function σ is strictly decreasing. For $\beta=1$, where $h(p)=p^{\alpha-1}$ and $H(p)=p^{\alpha}/\alpha$, it is easy to verify that σ displays the monotonicity properties illustrated in Fig. 1:

$$\sigma(p|\alpha,1)=1-\frac{\alpha+1}{\alpha}p \quad\text{and}\quad p_0=\frac{\alpha}{\alpha+1} \tag{4}$$

For Beta-distributions with $\beta<1$, Fig. 1 shows that every proposer-type $t_i>0$ has a unique value $p(t_i)$ that satisfies the necessary condition in (2). However, for the type $t_i=0$ there are *two* such values: $p=1$ and $p=p_0\in(0,1)$. To see that the optimal choice of the type $t_i=0$ is $p=p_0$ rather than $p=1$, note that this type's expected utility in (3) is zero for $p=1$, while it is $(1-p_0)H(p_0)/B(\alpha,\beta)>0$ for $p=p_0$. Therefore, Beta-distributions with $\beta>1$ also generate a unique optimal choice $p(t_i)$ for every proposer-type $t_i\in[0,1]$. □

Given the unique equilibrium of the DCM, a natural question is how the SCR implied by the DCM compares with a first best SCR. We address this question in Corollary 1 below, indicating

[4]Note that the specific value of p_0 depends on the distribution-parameters α and β.

the probability of the compromise which is the only component of the SCR that affects social welfare:

Corollary 1. *The SCR $\hat{f}$ implied by the equilibrium of the DCM in Prop. 1 features the following probability of the compromise:*

$$\hat{f}_0(t_1,t_2) = \begin{cases} 0 & \text{if } t_2 < \underline{s}(t_1) \\ 1/2 & \text{if } \underline{s}(t_1) < t_2 < \bar{s}(t_1) \\ 1 & \text{if } t_2 > \bar{s}(t_1) \end{cases} \tag{5}$$

where $\underline{s}(t_1) \equiv \min\{\sigma(t_1|\alpha,\beta), \sigma^{-1}(t_1|\alpha,\beta)\}$ *and* $\bar{s}(t_1) \equiv \max\{\sigma(t_1|\alpha,\beta), \sigma^{-1}(t_1|\alpha,\beta)\}$.

Proof. Observe that the function σ, when restricted to the subset $[0, p_0]$ of its domain, is invertible because it is strictly decreasing for all Beta-distributions. Denoting the inverse by σ^{-1}, we can express the proposer's optimal choice in item (i) of Prop. 1 as $p(t_i) = \sigma^{-1}(t_i|\alpha,\beta)$. Consequently, item (ii) of Prop. 1 can be re-stated as follows: the responder says 'yes' if $t_{-i} < p(t_i|\alpha,\beta) \Leftrightarrow t_i > \sigma(t_{-i}|\alpha,\beta)$; otherwise, the responder says 'no'. Since each agent has equal chance of being the proposer, we obtain the probability of the compromise in (5). $\square$

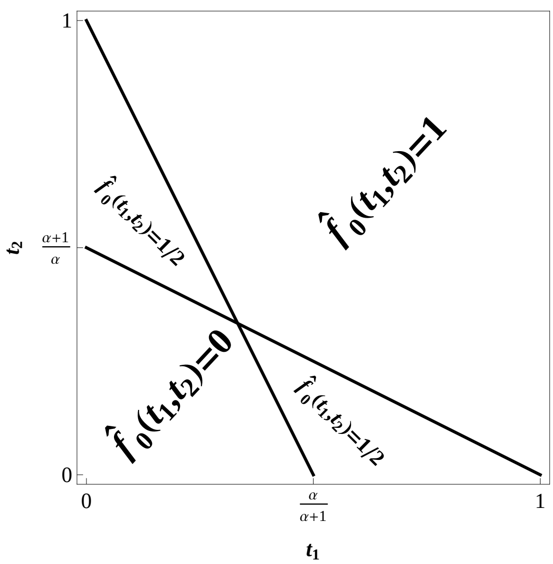

Figure 2: Prob. of compromise under DCM for uniform distribution

Fig. 2 provides an illustration of the probability of the compromise implied by the DCM when agents' types are drawn from the uniform distribution (i.e. $\alpha = \beta = 1$).[5] As there are type-pairs below the cross-diagonal in the unit square for which the DCM implements the compromise, it is easy to see that the probability in (5) differs markedly from the one associated with a first best SCR (as described in Section 2.3).

As it is difficult in general to characterize Beta-distributions for which the probability of the compromise implied by the DCM approximates first best, we focus in the remainder of this note on Beta-distributions with $\beta = 1$. The reason is that for these distributions, the probability of the compromise in (5) converges to first best for large α. By (4), $\lim_{\alpha\to\infty} \sigma(p|\alpha,1) = 1 - p$,

[5] In this case, by (4): $\sigma(p|1,1) = 1 - 2p$ and $p_0 = 1/2$. Similar figures, albeit with nonlinear σ, emerge for Beta-distributions with $\beta \neq 1$.

$\lim_{\alpha\to\infty} p_0 = 1$. Given this observation, we explore in the next section how ex ante welfare under the DCM changes with the distribution-parameter α.

3.2 Welfare

In this section, we study ex ante expected welfare under the DCM when the agents' types are drawn from a Beta-distribution with $\beta = 1$ and variable distribution-parameter α. There are two distinct advantages to working with this particular class of distributions. The first, already mentioned in the previous section, is that the probability of the compromise implied by the DCM converges to first best for large α. The second advantage is that we can compute analytically ex ante expected social welfare of the DCM. This is not possible for general Beta-distributions, for which one has to resort to numerical welfare-comparisons.

Note that because the two agents are ex ante symmetric and equally likely to be the proposer, we can compute ex ante welfare of the DCM under *any* Beta-distribution as an expectation over all type-pairs for which the compromise is chosen when agent 1 is the proposer. For the special case of Beta-distributions with $\beta = 1$, this yields the following expression, where Γ denotes the Gamma function:[6,7]

$$\begin{aligned}\hat{W}(\alpha,1) &= 1+\int_0^1\left(\int_{\frac{\alpha+1}{\alpha}(1-t_1)}^1 (t_1+t_2-1)\alpha t_2^{\alpha-1}dt_2\right)\alpha t_1^{\alpha-1}dt_1 \\ &= \frac{2\alpha\Gamma(\frac{1}{2}+\alpha)+\left(4+\frac{4}{\alpha}\right)^{-\alpha}\alpha\sqrt{\pi}\,\Gamma(\alpha)}{(1+\alpha)\Gamma(\frac{1}{2}+\alpha)} \qquad (6)\end{aligned}$$

Fig. 3 shows a graph of ex ante welfare $\hat{W}(\alpha,1)$ associated with the DCM. As a benchmark, the figure also depicts as a dashed curve the ex ante welfare under a first best SCR.[8] Note from Fig. 3 that ex ante welfare under the DCM converges to the first best welfare-level as the distribution-parameter α grows. This is not surprising, as the probability of the compromise implied by the DCM converges to first best as $\alpha \to \infty$. What appears more surprising is the non-monotonicity of $\hat{W}(\alpha,1)$, especially when noting that the probability of *not* choosing the compromise goes to zero under the DCM as α vanishes. Note, however, that for vanishing α, the type-distribution $H(t_i)/B(\alpha,\beta)$ concentrates ever more probability mass on very low realizations (i.e. types close to zero). This, in turn, leads to a negligibly small probability that type-pairs (t_1,t_2) materialize for which the compromise is chosen when it should not be.

We can conclude that for both very small and large α the chance of an inefficient collective choice by the DCM is very small. Note that relative to first best, a maximal ex ante welfare loss of 12.45% occurs when $\alpha \approx 0.174$. Under the uniform distribution ($\alpha = 1$), the relative welfare loss is 3.571%. This is clearly much larger than the welfare loss of 0.0191% generated by the *cropped triangle rule* in Börgers and Postl (2009). Note, however, that their characterization of the cropped triangle rule, especially the associated probabilities $f_i(t)$ that are needed to give the agents incentives for truthful revelation of their types, pertains exclusively to the case of

[6] Recall that $\Gamma(\alpha) \equiv \int_0^\infty e^{-x}x^{\alpha-1}dx$.

[7] We integrate over all type-pairs (t_1,t_2) s.t. $t_2 > \sigma^{-1}(t_1|\alpha,1) \equiv (\alpha+1)(1-t_1)/\alpha$, which captures all events in which agent 1 is the proposer and the compromise is chosen because agent 2 says 'no'. While it is straightforward to evaluate the inner integral in (6) w.r.t. t_2, we have used the 'Integrate'-function implemented in Mathematica 8.0.4 for Windows (64-bit) to compute analytically the outer integral w.r.t. t_1.

[8] First best welfare is $\left(\alpha^2\Gamma^2(\alpha)(1+\alpha)+2\alpha\Gamma(2\alpha+2)\right)/(1+\alpha)\Gamma(2\alpha+2)$, which we have verified using Mathematica.

the uniform distribution. We can therefore view the present results on the DCM as providing a lower bound on the welfare loss associated with any welfare-maximizing incentive compatible social choice rule for a much wider range of type-distributions.

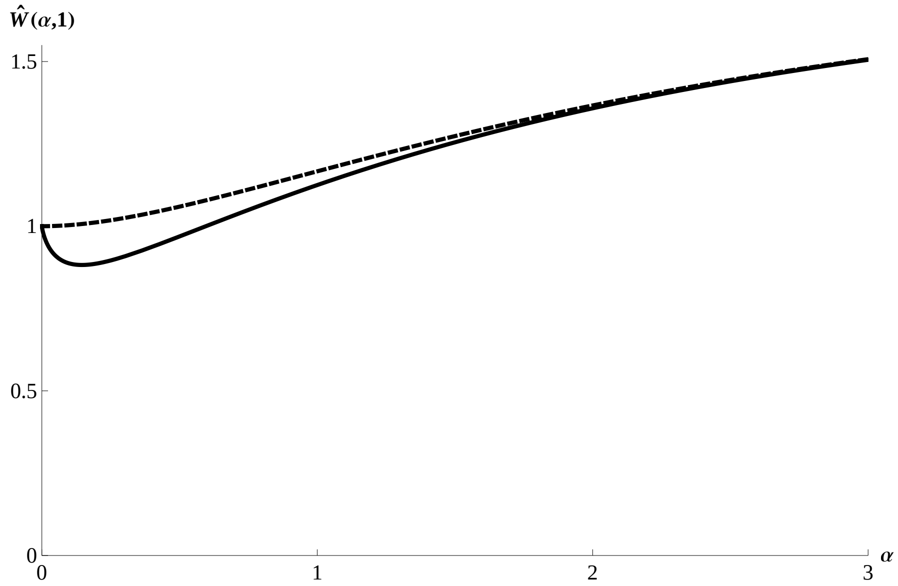

Figure 3: Welfare under DCM and first best (dashed) for Beta-distributions with $\beta = 1$

4 Appendix

The results presented in this appendix concern the monotonicity properties of the function $\sigma : [0,1] \to \mathbb{R}$, $p \mapsto \sigma(p|\alpha,\beta) = 1 - p - H(p)/h(p)$ associated with the parameterized Beta-distribution. We start by investigating the behavior of σ at the boundaries of its domain.

Lemma 1. *For all Beta-distributions, the function σ takes the value $\sigma(0|\alpha,\beta) = 1$ at $p = 0$.*

Proof. The value $\sigma(0|\alpha,\beta)$ depends on how the ratio $H(p)/h(p)$ behaves as $p \to 0$. We show in items (i)-(iii) below that $\lim_{p\to 0} H(p)/h(p) = 0$. While $\lim_{p\to 0} H(p) = 0$ for all Beta-distributions, the value of $\lim_{p\to 0} h(p)$ depends on the distribution parameter α:

(i) For $\alpha < 1$, $\lim_{p\to 0} h(p) = \infty$. We can write $H(p)/h(p)$ as $p^{1-\alpha}(1-p)^{1-\beta}H(p)$. The desired result follows because $\lim_{p\to 0} p^{1-\alpha} = 0$ and $\lim_{p\to 0}(1-p)^{1-\beta} = 1$.

(ii) For $\alpha = 1$, the result follows immediately since $\lim_{p\to 0} h(p) = 1$.

(iii) For $\alpha > 1$, $\lim_{p\to 0} h(p) = 0$. Using l'Hôpital's rule, we can establish: $\lim_{p\to 0} H(p)/h(p) = \lim_{p\to 0} h(p)/h'(p) = \lim_{p\to 0} p(1-p)/((\alpha-1)(1-p) - (\beta-1)p) = 0$. □

Lemma 2. *For all Beta-distributions, the function σ is nonpositive at $p = 1$.*

Proof. The value $\sigma(1|\alpha,\beta)$ depends on how the ratio $H(p)/h(p)$ behaves as $p \to 1$. We show in items (i)-(iii) below that $\lim_{p\to 1} H(p)/h(p) \geq 0$. While $\lim_{p\to 1} H(p) = B(\alpha,\beta)$, the value of $\lim_{p\to 1} h(p)$ depends on the distribution parameter β:

(i) For $\beta < 1$, $\lim_{p\to 1} h(p) = \infty$. As $H(p)/h(p) = p^{1-\alpha}(1-p)^{1-\beta}H(p)$, it follows immediately that $\lim_{p\to 1} H(p)/h(p) = 0$.

(ii) For $\beta = 1$, $\lim_{p\to 1} h(p) = 1$. Thus, $\lim_{p\to 1} H(p)/h(p) = B(\alpha,\beta) > 0$.

(iii) For $\beta > 1$, $\lim_{p\to 1} h(p) = 0$. Thus, $\lim_{p\to 1} H(p)/h(p) = \infty$. □

In the following Lemmas 3-5, we show formally that the monotonicity properties of σ indicated in Fig. 1 emerge for any Beta-distribution with $\beta > 1$, $\beta = 1$, and $\beta < 1$ resp. To give the reader a better sense of how the more general results will be established below, we start by considering the simplest case of Beta-distributions with $\alpha = 1$:

Lemma 3. *Consider the subclass of Beta-distributions with $\alpha = 1$: those with $\beta \geq 1$ generate a function σ that is strictly decreasing in p. For the remaining ones (i.e. those with $\beta < 1$), the function σ has a unique root $p_0 \in (0,1)$, with $\sigma(p|1,\beta) > 0$, $\sigma'(p|1,\beta) < 0$ for all $p \in [0,p_0)$, and $\sigma(p|1,\beta) < 0$ for all $p \in (p_0,1)$.*

Proof. For Beta-distributions with $\alpha = 1$, it is straightforward to compute:

$$\sigma(p|1,\beta) = 1 - p - \frac{1-(1-p)^{\beta}}{\beta(1-p)^{\beta-1}} \text{ and } \sigma'(p|1,\beta) = -2 + \frac{1-(1-p)^{\beta}}{\beta(1-p)^{\beta-1}} \cdot \frac{1-\beta}{1-p} \tag{A.1}$$

By distinguishing the cases $\beta \gtrless 1$, we can use the derivative σ' in (A.1) to establish the following additional properties of σ:[9]

I. For $\beta > 1$, $\lim_{p\to 1} \sigma'(p|1,\beta) = -\infty$, and $\sigma'(p|1,\beta) < 0$ for all $p \in [0,1)$.[10] This, together with item (ii) of Lemma 1 and item (iii) of Lemma 2 implies that σ has a unique root at some $p_0 \in (0,1)$.

II. For $\beta < 1$, σ has a unique turning point at $p_1 \equiv ((1-\beta)/(1+\beta))^{1/\beta}$: $\sigma'(p_1|1,\beta) = 0$, with $\sigma'(p|1,\beta) < 0$ for all $p < p_1$, and $\sigma'(p|1,\beta) > 0$ for all $p > p_1$. Furthermore, σ is negative at this point: $\sigma(p_1|1,\beta) < 0$.[11] This implies that σ has two roots: one at some point $p_0 \in (0,p_1)$, and the other at 1. To see this, note first that σ is strictly decreasing on $[0,p_1)$. Therefore, it has a single root p_0 within this interval. Next, note that since σ is strictly increasing on $(p_1,1]$ and $\sigma(1|1,\beta) = 0$, its sole root within this interval is $p = 1$. □

We now establish the monotonicity properties of the function σ for Beta-distributions with $\alpha,\beta \neq 1$. For this, we need both the first and second derivatives of σ. We state these derivatives here before presenting in Lemmas 4 and 5 our monotonicity results along with their proofs.[12] Note that we suppress for simplicity the dependence of σ on the distribution-parameters α and β. I.e. we write henceforth $\sigma(p)$ instead of $\sigma(p|\alpha,\beta)$.

$$\sigma'(p) = -2 + \lambda(p)\mu(p) \tag{A.2}$$

[9] For $\beta = 1$ the result in Lemma 3 is obvious from the expression for σ in (4).

[10] This follows because $(1-p)^{\beta} > -(\beta-1)/(\beta+1)$ for all p.

[11] To see this, use the fact that $\sigma'(p_1|1,\beta) = 0$ to obtain an expression for $(1-(1-p_1)^{\beta})/\beta(1-p_1)^{\beta-1}$. Then substitute this expression into $\sigma(p_1|1,\beta)$ in (A.1). This yields $\sigma(p_1|1,\beta) = -(1-p_1)(1+\beta)/(1-\beta) < 0$.

[12] We omit in the following all straightforward derivations for the sake of brevity.

where $\lambda(p) \equiv H(p)/h(p)$ and $\mu(p) \equiv (\alpha - 1 - p(\alpha - 1 + \beta - 1))/p(1-p)$. Observe that:[13]

$$\lim_{p\to 0} \sigma'(p) = -\frac{\alpha+1}{\alpha} \text{ and } \lim_{p\to 1} \sigma'(p) = \begin{cases} \infty & \text{if } \beta < 1 \\ -\infty & \text{if } \beta > 1 \end{cases} \tag{A.3}$$

Note that the first derivatives of the functions λ and μ are given by:

$$\lambda'(p) = 1 - \lambda(p)\mu(p) \text{ and } \mu'(p) = -\frac{(\alpha-1)(1-p)^2 + (\beta-1)p^2}{p^2(1-p)^2}$$

By differentiating (A.2) w.r.t. p and substituting in the expression for $\lambda'(p)$, we can write as follows the second derivative of σ:

$$\sigma''(p) = \lambda'(p)\mu(p) + \lambda(p)\mu'(p) = (1 - \lambda(p)\mu(p))\mu(p) + \lambda(p)\mu'(p) \tag{A.4}$$

Lemma 4. *For any Beta-distribution with $\alpha \neq 1$ and $\beta > 1$, the function σ is strictly decreasing.*

Proof. We show by contradiction that $\sigma'(p) < 0$ for all $p \in [0,1]$. We distinguish Beta-distributions with $\alpha < 1$ and $\alpha > 1$:

I. Let $\alpha < 1$. Suppose there is an interval $I \subset (0,1)$ s.t. $\sigma'(p) \geq 0$ for all $p \in I$. Then, for any $p \in I$ it must hold by (A.2) that: $\mu(p) = (-(1-\alpha) - p(-(1-\alpha) + \beta - 1))/p(1-p) > 0$, which is equivalent to the statement: $1 - \alpha > \beta - 1$ and $p > (1-\alpha)/(1-\alpha-(\beta-1)) > 1$. This constitutes a contradiction as $p < 1$ for all $p \in I$.

II. Let $\alpha > 1$. Observe from (A.2) and (A.3) that $\sigma'(0) < 0$ and $\sigma'(p) \leq -2$ for all $p \in [\bar{p}, 1]$, where $\bar{p} \equiv (\alpha-1)/(\alpha-1+\beta-1)$. Now suppose there exists an interval $(l,u) \subset (0,\bar{p})$ s.t. $\sigma'(p) > 0$ for all $p \in (l,u)$. Then there is a turning point $p_1 < l$ s.t. σ' changes sign from '$-$' to '$+$': $\sigma'(p_1) = 0$ and $\sigma''(p_1) > 0$. This leads to a contradiction: to see from (A.4) that $\sigma''(p_1) < 0$, note that: (i) $\sigma'(p_1) = 0 \Leftrightarrow \lambda(p_1)\mu(p_1) = 2$ (this follows from (A.2)); (ii) $\mu(p_1) > 0$ as $p_1 < \bar{p}$; and (iii) $\mu'(p_1) < 0$. □

Lemma 5. *For any Beta-distribution with $\alpha \neq 1$ and $\beta < 1$, the function σ has a unique root $p_0 \in (0,1)$, with $\sigma(p) > 0$, $\sigma'(p) < 0$ for all $p \in [0, p_0)$, and $\sigma(p) < 0$ for all $p \in (p_0, 1)$.*

Proof. Recall that for $\beta < 1$: $\sigma(0) = 1$, $\sigma'(0) = -(\alpha+1)/\alpha$, $\sigma(1) = 0$, and $\lim_{p\to 1} \sigma'(p) = \infty$. These properties imply that σ must cross the horizontal axis at least once, and that there is an interval $(p_1, 1]$ on which σ is increasing as it approaches 0 from below for $p \to 1$. Thus, σ must have a turning point p_1, with $\sigma(p_1) < 0$, at which the derivative σ' changes sign from '$-$' to '$+$'. We argue by contradiction that p_1 is the *unique* stationary point of σ. To do this, we show that if there are multiple stationary points, then σ' must change sign from '$-$' to '$+$' at *each one* of them: $\sigma''(\hat{p}) > 0$ for every stationary point $\hat{p}$. This leads to the desired contradiction, because if σ' changes sign from '$-$' to '$+$' at a stationary point p_1, it cannot again change sign from '$-$' to '$+$' at another stationary point p_2 without previously having changed sign from '$+$' to '$-$' at a stationary point between p_1 and p_2. To see that $\sigma''(\hat{p}) > 0$ for every stationary point $\hat{p}$, we distinguish Beta-distributions with $\alpha < 1$ and $\alpha > 1$:

[13] The first limit is obtained by L'Hôpital's rule: $\lim_{p\to 0} H(p)/h(p)p(1-p) = \lim_{p\to 0} 1/(\alpha(1-p) - \beta p) = 1/\alpha$; the second limit follows as $\lim_{p\to 1} H(p)/h(p)p(1-p) = \infty$.

I. Let $\alpha < 1$. Note from (A.2) that if $\hat{p}$ is a stationary point, then $\mu(\hat{p}) > 0$. Also, $\sigma'(\hat{p}) = 0 \Leftrightarrow \lambda(\hat{p}) = 2/\mu(\hat{p})$. Therefore, the second derivative of σ in (A.4) becomes:

$$\sigma''(\hat{p}) = -\mu(\hat{p}) + 2\left((1-\alpha)(1-\hat{p})^2 + (1-\beta)\hat{p}^2\right)/\mu(\hat{p})\hat{p}^2(1-\hat{p})^2 \qquad \text{(A.5)}$$

We now show indirectly that $\sigma''(\hat{p}) > 0$. Suppose instead that $\sigma''(\hat{p}) \leq 0$. By (A.5):

$$2\left((1-\alpha)(1-\hat{p})^2 + (1-\beta)\hat{p}^2\right)/\hat{p}^2(1-\hat{p})^2 \leq (\mu(\hat{p}))^2$$

Substituting in the explicit expression for $\mu(\hat{p})$ given above, we can state after straightforward manipulation that $\sigma''(\hat{p}) \leq 0$ iff:

$$(1-\alpha)(1-\hat{p})^2(1+\alpha) + (1-\beta)\hat{p}^2(1+\beta) \leq -2(1-\alpha)(1-\beta)\hat{p}(1-\hat{p})$$

The latter inequality involves a contradiction because the left-hand side is positive, while the right-hand side is negative. We can therefore conclude that $\sigma''(\hat{p}) > 0$ for all stationary points $\hat{p}$, as required.

II. Let $\alpha > 1$. Note first that $\mu(p) > 0$ for all $p \in [0,1]$. Now consider a turning point $\hat{p}$ at which σ changes sign from '$-$' to '$+$'. I.e. $\sigma''(\hat{p}) > 0$, with:

$$\sigma''(\hat{p}) = -\mu(\hat{p}) - 2\left((\alpha-1)(1-\hat{p})^2 + (\beta-1)\hat{p}^2\right)/\mu(\hat{p})\hat{p}^2(1-\hat{p})^2$$

Substituting in the explicit expression for $\mu(\hat{p})$ given above, we can state after straightforward manipulation that $\sigma''(\hat{p}) > 0$ iff:

$$\left((\alpha-1)(1-\hat{p})^2 + (\beta-1)\hat{p}^2\right)/(a-1-\hat{p}(a-1+b-1))^2 < -1/2 \qquad \text{(A.6)}$$

Note that the ratio on the left-hand side of (A.6) is a monotonically decreasing function of p.[14] This implies that (A.6) also holds for any stationary point $\tilde{p} > \hat{p}$, and therefore $\sigma''(\tilde{p}) > 0$. As $\sigma'(0) < 0$, we know that the first sign-change of the derivative σ' is from '$-$' to '$+$'. Therefore, we must have $\sigma''(\hat{p}) > 0$ for *all* stationary points $\hat{p}$ of σ. $\square$

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[14] The ratio takes value $1/(\alpha-1)$ at $p=0$, and $-1/(1-\beta)$ at $p=1$. Its derivative is negative as it features a positive denominator, and a numerator given by: $-2(\alpha-1)(1-\beta)((\alpha-1)(1-p)+(1-\beta)p) < 0$.

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